SCHMIDT THEORY FOR STIRLING ENGINES

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1. INTRODUCTION

The Schmidt theory is one of the isothermal calculation methods for Stirling engines. It is the most simple method and very useful during Stirling engine development.

This theory is based on the isothermal expansion and compression of an ideal gas.

2. ASSUMPTION OF SCHMIDT THEORY

The performance of the engine can be calculated a P-V diagram. The volume in the engine is easily calculated by using the internal geometry. When the volume, mass of the working gas and the temperature is decided, the pressure is calculated using an ideal gas method as shown in equation (1).

$$PV = mRT \tag{1}$$

The engine pressure can be calculated under following assumptions:

(a) There is no pressure loss in the heat exchangers and there are no internal pressure differences.
(b) The expansion process and the compression process changes isothermal.
(c) Conditions of the working gas are changed as an ideal gas.
(d) There is a perfect regeneration.
(e) The expansion dead space maintains the expansion gas temperature - T_E, the compression dead space maintains the compression gas temperature - T_C during the cycle.
(f) The regenerator gas temperature is an average of the expansion gas temperature - T_E and the compression gas temperature - T_C.
(g) The expansion space - V_E and the compression space - V_C changes according sine curves.

Table 1 shows symbols used the Schmidt Theory.

3. ALPHA-TYPE STIRLING ENGINE

Figure 1 shows the calculation model of Alpha-type Stirling engine.

The volumes of the expansion- and compression cylinder at a given crank angle are determined at first. The momental volumes is described with a crank angle - x. This crank angle is defined as x=0 when the expansion piston is located the most top position (top dead point).

The momental expansion volume - V_E is described in equation (2) with a swept volume of the expansion piston - V_{SE}, an expansion dead volume - V_{DE} under the condition of assumption (g).

$$V_E = \frac{V_{SE}}{2}(1-\cos x)+V_{DE} \qquad (2)$$

The momental compression volume - V_C is found in equation (3) with a swept volume of the compression piston - V_{SC}, a compression dead volume - V_{DC} and a phase angle - dx.

$$V_C = \frac{V_{SC}}{2}\{1-\cos(x-dx)\}+V_{DC} \qquad (3)$$

The total momental volume is calculated in equation (4).

$$V = V_E + V_R + V_C \qquad (4)$$

By the assumptions (a), (b) and (c), the total mass in the engine - m is calculated using the engine pressure - P, each temperature - T, each volume - V and the gas constant - R.

$$m = \frac{PV_E}{RT_E} + \frac{PV_R}{RT_R} + \frac{PV_C}{RT_C} \qquad (5)$$

The temperature ratio - t, a swept volume ratio - v and other dead volume ratios are found using the following equations.

$$t = \frac{T_C}{T_E} \qquad (6)$$

$$v = \frac{V_{SC}}{V_{SE}} \qquad (7)$$

$$X_{DE} = \frac{V_{DE}}{V_{SE}} \qquad (8)$$

$$X_{DC} = \frac{V_{DC}}{V_{SE}} \qquad (9)$$

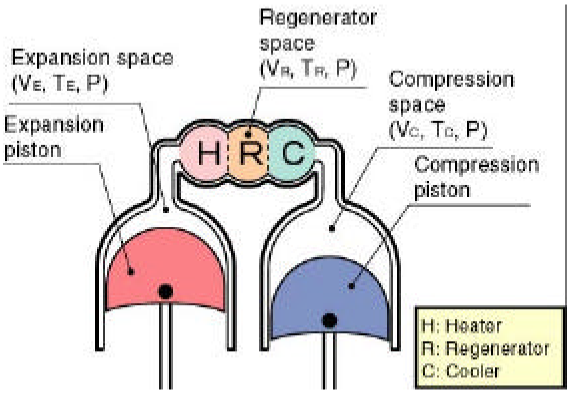

Fig. 1 Alpha-type Stirling Engine

Table 1 Symbols

Name	Symbol	Unit
Engine pressure	P	Pa
Swept volume of expansion piston or displacer piston	V_{SE}	m^3
Swept volume of compression piston or power piston	V_{SC}	m^3
Dead volume of expansion space	V_{DE}	m^3
Regenerator volume	V_R	m^3
Dead volume of compression space	V_{DC}	m^3
Expansion space momental volume	V_E	m^3
Compression space momental volume	V_C	m^3
Total momental volume	V	m^3
Total mass of working gas	m	kg
Gas constant	R	J/kgK
Expansion space gas temperature	T_H	K
Compression space gas temprature	T_C	K
Regenerator space gas temperature	T_R	K
Phase angle	dx	deg
Temperatuer ratio	t	
Swept volume ratio	v	
Dead volume ratio	X	
Engine speed	n	Hz
Indicated expansion energy	W_E	J
Indicated compression energy	W_C	J
Indicated energy	W_i	J
Indicated expansion power	L_E	W
Indicated compression power	L_C	W
Indicated power	L_i	W
Indicated efficiency	e	

$$X_R = \frac{V_R}{V_{SE}} \tag{10}$$

The regenerator temperature - T_R is calculated in equation (11), by using the assumption (f).

$$\mathrm{T_R} = \frac{\mathrm{T_E} + \mathrm{T_C}}{2} \tag{11}$$

When equation (5) is changed using equation (6)-(10), the total gas mass - m is described in the next equation.

$$m = \frac{P}{RT_C}(tV_E + \frac{2tV_R}{1+t} + V_C) \tag{12}$$

Equation (12) is changed in equation (13), using equation (2) and (3).

$$m = \frac{PV_{SE}}{2RT_C}\{S - B\cos(x-a)\} \tag{13}$$

Now;

$$a = \tan^{-1}\frac{v \cdot \sin dx}{t + \cos dx} \tag{14}$$

$$S = t + 2tX_{DE} + \frac{4tX_R}{1+t} + v + 2X_{DC} \tag{15}$$

$$B = \sqrt{t^2 + 2tv\cos dx + v^2} \tag{16}$$

The engine pressure - P is defined as the next equation using equation (13).

$$P = \frac{2mRT_C}{V_{SE}\{S - B\cos(q - a)\}} \tag{17}$$

The mean pressure - P_{mean} can be calculated as follows:

$$P_{mean} = \frac{1}{2p}\oint P dx = \frac{2mRT_C}{V_{SE}\sqrt{S^2 - B^2}} \tag{18}$$

c is defined in the next equation.

$$c = \frac{B}{S} \tag{19}$$

As a result, the engine pressure - P, based the mean engine pressure - P_{mean} is calculated in equation (20).

$$P = \frac{P_{mean}\sqrt{S^2 - B^2}}{S - B\cos(x-a)} = \frac{P_{mean}\sqrt{1-c^2}}{1 - c \cdot \cos(x-a)} \tag{20}$$

On the other hand, in the case of equation (17), when $\cos(x\text{-}a)$=-1, the engine pressure - P becomes the minimum pressure - P_{min}, the next equation is introduced.

$$P_{\min} = \frac{2mRT_C}{V_{SE}(S+B)} \tag{21}$$

Therefore, the engine pressure - P, based the minimum pressure - P_{min} is described in equation (22).

$$P = \frac{P_{\min}(S+B)}{S - B\cos(x-a)} = \frac{P_{\min}(1+c)}{1 - c \cdot \cos(x-a)} \tag{22}$$

Similarly, when $\cos(x\text{-}a)$=1, the engine pressure - P becomes the maximum pressure - P_{max}. The following equation is introduced.

$$P = \frac{P_{\max}(S-B)}{S-B\cos(x-a)} = \frac{P_{\max}(1-c)}{1-c\cdot\cos(x-a)} \tag{23}$$

The P-V diagram of Alpha-type Stirling engine can be made with above equations.

4. BETA-TYPE STIRLING ENGINE

Similarly, the equations for Beta-type Stirling engine are declared. Figure 2 shows a calculation model of a Beta-type Stirling engine. The expansion momental volume - V_E and the compression momental volume - V_C are described in the following equations, with a swept volume of a displacer piston - V_{SE}, a swept volume of a power piston - V_{SC} and a phase angle -dx between the displacer piston and power piston.

$$V_E = \frac{V_{SE}}{2}(1-\cos x) + V_{DE} \tag{24}$$

$$V_C = \frac{V_{SE}}{2}(1+\cos x) + \frac{V_{SC}}{2}\{1-\cos(x-dx)\} + V_{DC}$$

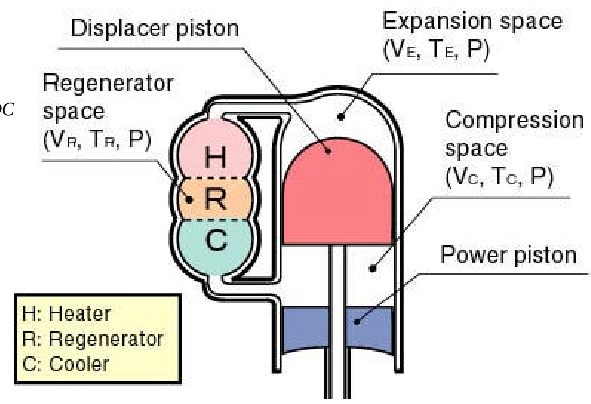

Fig. 2 Beta-type Stirling Engine

In the case of the Beta-type Stirling engine, the displacer piston and the power piston are located in the same cylinder. When both pistons overlap their stroke, an effective working space is created. The overlap volume - V_B in equation (25) can be calculated in the next equation.

$$V_B = \frac{V_{SE}+V_{SC}}{2} - \sqrt{\frac{{V_{SE}}^2+{V_{SC}}^2}{4} - \frac{V_{SE}V_{SC}}{2}\cos dx} \tag{26}$$

Then the total momental volume - V is found in equation (27).

$$V = V_E + V_R + V_C \tag{27}$$

The engine pressure - P based the mean pressure - P_{mean}, the minimum pressure - P_{min} and the maximum pressure - P_{max} are described in the following equations like the Alpha-type Stirling engine.

$$P = \frac{P_{mean}\sqrt{1-c^2}}{1-c\cdot\cos(x-a)} = \frac{P_{\min}(1+c)}{1-c\cdot\cos(x-a)} = \frac{P_{\max}(1-c)}{1-c\cdot\cos(x-a)} \tag{28}$$

Several ratios and coefficients are defined as follows.

$$t = \frac{T_C}{T_E} \tag{29}$$

$$v = \frac{V_{SC}}{V_{SE}} \tag{30}$$

$$X_B = \frac{V_B}{V_{SE}} \quad (31)$$

$$X_{DE} = \frac{V_{DE}}{V_{SE}} \quad (32)$$

$$X_{DC} = \frac{V_{DC}}{V_{SE}} \quad (33)$$

$$X_R = \frac{V_R}{V_{SE}} \quad (34)$$

$$a = \tan^{-1} \frac{v \sin dx}{t + \cos dx + 1} \quad (35)$$

$$S = t + 2tX_{DE} + \frac{4tX_R}{1+t} + v + 2X_{DC} + 1 - 2X_B \quad (36)$$

$$B = \sqrt{t^2 + 2(t-1)v\cos dx + v^2 - 2t + 1} \quad (37)$$

$$c = \frac{B}{S} \quad (38)$$

The P-V diagram of Beta-type Stirling engine can be made with above equations.

5. GAMMA-TYPE STIRLING ENGINE

Figure 3 shows a calculation model of a Gamma-type Stirling engine.

Similar calculation equations are made as the Alpha- and Beta-type engine. The expansion momental volume - V_E and the compression momental volume - V_C are described in the following equations with a swept volume of a displacer piston - V_{SE}, a swept volume of a power piston - V_{SC} and a phase angle - dx between the displacer piston and the power piston..

$$V_E = \frac{V_{SE}}{2}(1 - \cos x) + V_{DE} \quad (39)$$

$$V_C = \frac{V_{SE}}{2}(1 + \cos x) + \frac{V_{SC}}{2}\{1 - \cos(x - dx)\} + V_{DC}$$

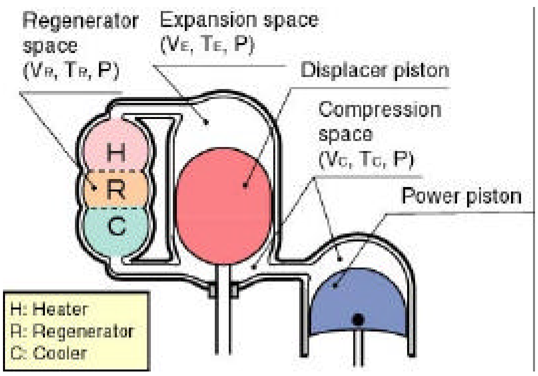

Fig. 3 Gamma-type Stirling Engine

The total momental volume - V is described the next equation.

$$V = V_E + V_R + V_C$$

The engine pressure - P based the mean pressure - P_{mean}, the minimum pressure - P_{min} and the maximum pressure - P_{max} are found in the following equations.

$$P = \frac{P_{mean}\sqrt{1-c^2}}{1 - c\cdot\cos(x-a)} = \frac{P_{\min}(1+c)}{1 - c\cdot\cos(x-a)} = \frac{P_{\max}(1-c)}{1 - c\cdot\cos(x-a)} \quad (42)$$

Now,

$$t = \frac{T_C}{T_E} \tag{43}$$

$$v = \frac{V_{SC}}{V_{SE}} \tag{44}$$

$$X_{DE} = \frac{V_{DE}}{V_{SE}} \tag{45}$$

$$X_{DC} = \frac{V_{DC}}{V_{SE}} \tag{46}$$

$$X_R = \frac{V_R}{V_{SE}} \tag{47}$$

$$a = \tan^{-1} \frac{v \sin dx}{t + \cos dx - 1} \tag{48}$$

$$S = t + 2tX_{DE} + \frac{4tV_R}{1+t} + v + 2X_{DC} + 1 \tag{49}$$

$$B = \sqrt{t^2 + 2(t-1)v\cos dx + v^2 - 2t + 1} \tag{50}$$

$$c = \frac{B}{S} \tag{51}$$

The P-V diagram of Gamma-type Stirling engine can be made with above equations.

6. INDICATED ENERGY, POWER AND EFFICIENCY

The indicated energy (area of the P-V diagram) in the expansion and compression space can be calculated as an analytical solution with use of the above coefficients. The indicated energy in the expansion space (indicated expansion energy) - W_E(J), based on the mean pressure - P_{mean}, the minimum pressure - P_{min} and the maximum pressure - P_{max} are described in the following equations.

$$W_E = \oint P dV_E = \frac{P_{mean} V_{SE} pc \cdot \sin a}{1+\sqrt{1-c^2}} = \frac{P_{\min} V_{SE} pc \cdot \sin a}{1+\sqrt{1-c^2}} \cdot \frac{\sqrt{1+c}}{\sqrt{1-c}} = \frac{P_{\max} V_{SE} pc \cdot \sin a}{1+\sqrt{1-c^2}} \cdot \frac{\sqrt{1-c}}{\sqrt{1+c}} \tag{52}$$

The indicated energy in the compression space (indicated compression energy) - W_C(J) are described in the next equations.

$$W_C = \oint P dV_C = -\frac{P_{mean} V_{SE} pct \cdot \sin a}{1+\sqrt{1-c^2}} = -\frac{P_{\min} V_{SE} pct \cdot \sin a}{1+\sqrt{1-c^2}} \cdot \frac{\sqrt{1+c}}{\sqrt{1-c}} = -\frac{P_{\max} V_{SE} pct \cdot \sin a}{1+\sqrt{1-c^2}} \cdot \frac{\sqrt{1-c}}{\sqrt{1+c}} \tag{53}$$

The indicated energy per one cycle of this engine - W_i(J) is demanded in the next equations.

$$W_i = W_e + W_c$$
$$= \frac{P_{mean} V_{SE} pc(1-t)\sin a}{1+\sqrt{1-c^2}} = \frac{P_{\min} V_{SE} pc(1-t)\sin a}{1+\sqrt{1-c^2}} \cdot \frac{\sqrt{1+c}}{\sqrt{1-c}} = \frac{P_{\max} V_{SE} pc(1-t)\sin a}{1+\sqrt{1-c^2}} \cdot \frac{\sqrt{1-c}}{\sqrt{1+c}} \tag{54}$$

Relations between P_{mean}, P_{min} and P_{max} are determined in the following equations.

$$\frac{P_{\min}}{P_{mean}} = \sqrt{\frac{1-c}{1+c}} \tag{55}$$

$$\frac{P_{\max}}{P_{mean}} = \sqrt{\frac{1+c}{1-c}} \tag{56}$$

The indicated expansion power - L_E(W), the indicated compression power - L_C(W) and the indicated power of this engine - L_i(W) are defined in the following equations, using the engine speed per one second , n(rps, Hz).

$$L_E = W_E n \tag{57}$$

$$L_C = W_C n \tag{58}$$

$$L_i = W_i n \tag{59}$$

The indicated expansion energy - W_E found equation (52) means an input heat from a heat source to the engine. The indicated compression energy - W_c calculated by equation (53) means a reject heat from the engine to cooling water or air. Then the thermal efficiency of the engine - e is calculated in the next equation.

$$e = \frac{W_i}{W_E} = 1 - t \tag{60}$$

This efficiency equals that of a Cornot cycle that is the most highest efficiency in every thermal engine.

7. EXAMPLE OF CALCULATION

EXERCISE:

Make a P-V diagram and calculate the indicated power of an Alpha-type Stirling engine under following conditions.

Swept volume of an expansion piston: 0.628 cm^3, swept volume of a compression piston: 0.628 cm^3, dead volume of the expansion space: 0.2cm^3, dead volume of the compression space: 0.2cm^3, regenerator volume: 0.2cm^3, phase angle: 90deg, mean pressure: 101.3 kPa, expansion gas temperature: 400degC, compression gas temperature: 30degC, engine speed: 2000 rpm.

A temperature ratio - t, a swept volume ratio - v and other dead volume ratio are calculated with the equation (6) - (10).

$$t = \frac{30+273}{400+273} = 0.450$$

$$v = \frac{0.628\times10^{-6}}{0.628\times10^{-6}} = 1.000$$

$$X_{DE} = \frac{0.2\times10^{-6}}{0.628\times10^{-6}} = 0.318$$

$$X_{DC} = \frac{0.2\times10^{-6}}{0.628\times10^{-6}} = 0.318$$

$$X_R = \frac{0.2\times10^{-6}}{0.628\times10^{-6}} = 0.318$$

Each coefficient is calculated with equation (14) - (16) and (19).

$$a = \tan^{-1}\frac{1\times\sin 90^\circ}{0.45+\cos 90^\circ} = 65.772^\circ$$

$$S = 0.45 + 2\times0.45\times0.318 + \frac{4\times0.45\times0.318}{1+0.450} + 1 + 2\times0.318 = 2.767$$

$$B = \sqrt{0.45^2 + 2\times0.45\times\cos\frac{p}{2} + 1} = 1.097$$

$$c = \frac{1.097}{2.767} = 0.396$$

Engine pressure is calculated with equation (20).

When crank angle - x=0 deg:

$$P = \frac{101.3\times10^3\sqrt{1-0.396^2}}{1-0.396\cos(0-65.772)} = 101.988\times10^3(Pa) = 101.988(kPa)$$

Similarly, when x=10 deg:

$$P = 109.893(kPa)$$

When x=20deg:

$$P = 118.011(kPa)$$

…

Next each momental volume is calculated with equation (2) ~ (4).

When crank angle, x=0 deg:

$$V_E = \frac{0.628\times10^{-6}}{2}(1-\cos 0^\circ) + 0.2 = 0.200\times10^{-6}(m^3) = 0.200(cm^3)$$

$$V_C = \frac{0.628\times10^{-6}}{2}\{1-\cos(0^\circ-90^\circ)\} + 0.2 = 0514\times10^{-6}(m^3) = 0.514(cm^3)$$

$$V = 0.2+0.2+0.514 = 0.914(cm^3)$$

When x=10 deg:

$$V = 0.864(cm^3)$$

When x=20 deg:

$$V = 0.826(cm^3)$$

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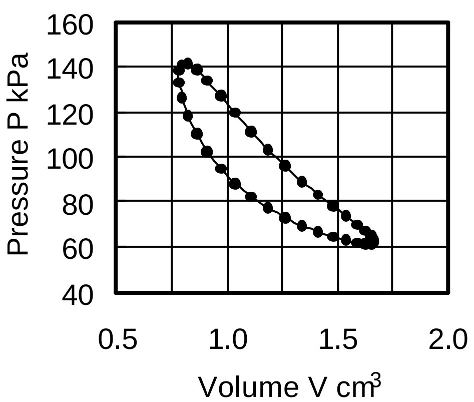

Fig. 4 P-V Diagram

Repeat above calculation to one complete cycle and plot the volumes - V and pressures - P on a graph paper.

An example of the P-V diagram is shown in Fig. 4.

The indicated energy is calculated with equation (52), (53) and (54).

$$W_E = \frac{101.3\times10^3\times0.628\times10^{-6}\times3.14\times0.396\times\sin 65.772^\circ}{1+\sqrt{1-0.396^2}} = 3.760\times10^{-2}(J)$$

$$W_C = -\frac{101.3\times10^3\times0.628\times10^{-6}\times3.14\times0.396\times0.45\times\sin 65.772^\circ}{1+\sqrt{1-0.396^2}} = -1.692\times10^{-2}(J)$$

$$W_i = 3.760\times10^{-2} - 1.692\times10^{-2} = 2.068\times10^{-2}(J)$$

The indicated power of the engines is calculated with equation (59).

$$\mathrm{L_i} = \frac{5.452\times10^{-2}\times2000}{60} = 0.689(\mathrm{W})$$

The indicated power of this engine is 0.689 W.

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REFFERENCES

1) G. Walker., Stirling Engines, (1980),17, Oxford Univ. Press.