Skin friction and pressure: the "footprints" of turbulence

Thomas R. Bewley and Bartosz Protas
Dept. of MAE, UC San Diego, La Jolla, CA

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This article addresses the information available at the wall in the problem of state estimation in wall-bounded incompressible flows. It is shown that, if precise measurements are made of the two components of wall skin friction, $\partial u/\partial y$ and $\partial w/\partial y$, and the wall pressure, p, all terms in the Taylor-series expansions of the flow state near the wall may be determined. Combining this fact with the analyticity of solutions of the Navier-Stokes equation on the attractor, in theory complete reconstruction of a turbulent flow in a channel at time t is possible given only precise measurements of the flow at the wall in a neighborhood of time t. Implications of this result, in light of the standard framework for adjoint-based state reconstruction in turbulent flow systems, are discussed.

1 Introduction

During the last 10 years, there has been a flurry of activity in controlling both laminar and turbulent flows in certain idealized settings. The goal of this research thrust has been twofold: to learn more about fundamental flow physics, and to begin to shed light on how to control fluid flow in practical engineering applications with model-based control strategies. For recent surveys of this active field of research, see, e.g., Gad-el-Hak (2001) and Bewley (2001), and the references contained therein.

An important and largely unsolved problem in model-based feedback control of turbulence is the estimation of the flow state based on the available flow measurements. From the literature survey we have performed (see the above-mentioned review articles for several examples), it appears that, to date, all efforts to control and/or estimate wall-bounded flows with information available at the wall only have used measurements of *either* wall skin friction[1] *or* wall pressure. The purpose of the present note is to show that much more complete information about the state of the system is available if measurements of both components of wall skin friction *and* the wall pressure are used.

In §2, it is shown that, if precise measurements are made of the two components of wall skin friction, $\partial u/\partial y$ and $\partial w/\partial y$, and the wall pressure, p, an arbitrary number of terms in the Taylor-series expansions of the flow state near the wall may be determined. In §3, it is shown using a high-fidelity DNS database of an $Re_\tau = 180$ turbulent channel flow that higher-order terms in truncated Taylor-series expansions uniformly improve the quality of the static reconstruction of the turbulent flow state near the wall when accurate measurements at the wall are available.

In practice, measurements are noisy, and dynamic estimation of the state, such as Riccati-based extended Kalman filters and adjoint-based methods for model predictive estimation, are much better behaved than Taylor-series expansions for the purpose of estimating the state based on noisy measurements. Such techniques assimilate the information contained in the available measurements into the estimate of the state without differentiation of the measurements. In §4, algorithms are presented by which all three types of available wall measurements may be accounted for in these types of state estimation strategies.

1.1 Governing equations

The present paper considers the problem of incompressible flow in a channel with known Dirichlet boundary conditions on the velocity field at the walls, $\{u_w, v_w, w_w\}$, known forcing $\{F_1, F_2, F_3\}$ on the interior of the flow, and known measurements of the skin-friction and pressure distributions on the walls, $\{M_1 \triangleq \frac{\partial u}{\partial y}|_w, M_2 \triangleq p|_w, M_3 \triangleq \frac{\partial w}{\partial y}|_w\}$. Initial conditions on the flow are unknown; we desire to reconstruct (or estimate) the flow in the channel based on the other information which is available.

Without loss of generality, §2 and 3 analyze the region adjacent to one of the walls, defining the $x-y-z$ coordinate system such that y is the wall-normal direction, with the wall located at $y = 0$. In the sections that follow §3, we switch to an $x_1 - x_2 - x_3$ coordinate system, and consider the flow in the entire channel $(0 \times L_1) \times (-1 \times 1) \times (0 \times L_3)$.

[1] Note that referring to the boundary values of $\partial u/\partial y$ and $\partial w/\partial y$ as "wall skin friction" is, admittedly, a bit sloppy notationally, as the corresponding components of the shear-stress tensor at the wall, $\tau_{xy} = \mu(\partial u/\partial y + \partial v/\partial x)$ and $\tau_{zy} = \mu(\partial w/\partial y + \partial v/\partial z)$, both include contributions from the (prescribed) boundary values of v on the wall and are scaled by the viscosity μ. We assume the viscosity μ and the value of v at the wall are known in this work, so $\partial u/\partial y$ and $\partial w/\partial y$ may easily be determined from measurements of τ_{xy} and τ_{zy} at the wall. The idealized problem of a continuous distribution of both actuation and sensing on the wall is not quite physically realizable anyway; how this configuration might be approximated in a real implementation is an application-specific issue which we will not address here. We will thus use the words "streamwise and spanwise wall skin friction distributions" to refer to the distributions of $\partial u/\partial y$ and $\partial w/\partial y$ on the wall without ambiguity, with apology to the reader for this abuse of notation.

The Navier-Stokes equation governing the flow is given by

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} - \frac{\partial p}{\partial x} + \nu\Delta u + F_1, \tag{1a}$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - w\frac{\partial v}{\partial z} - \frac{\partial p}{\partial y} + \nu\Delta v + F_2, \tag{1b}$$

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} - v\frac{\partial w}{\partial y} - w\frac{\partial w}{\partial z} - \frac{\partial p}{\partial z} + \nu\Delta w + F_3, \tag{1c}$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{2}$$

where $\Delta \triangleq \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. The continuity equation (2) constrains the three velocity components $\{u, v, w\}$, which evolve according to the momentum equations (1a)-(1c), to lie in a divergence-free subspace. This constraint is applied through the influence of the pressure p in the momentum equations, which acts as a Lagrange multiplier in these three equations in such a way that the continuity equation is satisfied at every point in space and every instant in time. We thus see that the Navier-Stokes equation effectively admits only *two degrees of freedom per spatial location*. Noting this fact, it is common to represent solutions to incompressible Navier-Stokes systems in a reduced, divergence-free form, thus applying the continuity equation implicitly.

One popular divergence-free form, convenient in terms of the imposition of Dirichlet boundary conditions on the velocity at walls, is the "v-ω_y" form, in which the wall-normal component of velocity, v, and the wall-normal component of vorticity, $\omega_y \triangleq \partial u/\partial z - \partial w/\partial x$, are retained as the two independent degrees of freedom per spatial location. From these two fields and the appropriate boundary conditions, u and w may be reconstructed exactly, and p may be determined up to an arbitrary constant. In the v-ω_y formulation, evolution equations governing v and ω_y are found by appropriate manipulation of (1) and (2). The right-hand sides of these equations may be interpreted as functions of v and ω_y only by substitution of the appropriate formulae for the reconstructions of u, w, and p.

The fact that the variables u, v, w, and p are not all independent in incompressible flows can easily lead to the mistaken impression that wall measurements of $\partial u/\partial y$, $\partial w/\partial y$, and p must in some sense be redundant. The purpose of the present note is to dispel this mistaken impression. To do this, we will show that the complete Taylor-series expansions of the velocity, vorticity, and pressure fields can be obtained from the three available wall measurements, though these expansions must be truncated at extremely low order if any of these three measurements is omitted.

2 Taylor-Series expansions of velocity, vorticity, and pressure

2.1 The general case

The Taylor-series expansions at the wall of the individual components of the velocity and vorticity vectors and the pressure may be written the following form:

$$u(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n u}{\partial y^n}\Big|_w y^n \triangleq \sum_{n=0}^{\infty} \frac{1}{n!} a_n(x,z,t)\, y^n, \qquad \omega_x(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n \omega_x}{\partial y^n}\Big|_w y^n \triangleq \sum_{n=0}^{\infty} \frac{1}{n!} e_n(x,z,t)\, y^n,$$

$$v(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n v}{\partial y^n}\Big|_w y^n \triangleq \sum_{n=0}^{\infty} \frac{1}{n!} b_n(x,z,t)\, y^n, \qquad \omega_y(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n \omega_y}{\partial y^n}\Big|_w y^n \triangleq \sum_{n=0}^{\infty} \frac{1}{n!} f_n(x,z,t)\, y^n,$$

$$w(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n w}{\partial y^n}\Big|_w y^n \triangleq \sum_{n=0}^{\infty} \frac{1}{n!} c_n(x,z,t)\, y^n, \qquad \omega_z(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n \omega_z}{\partial y^n}\Big|_w y^n \triangleq \sum_{n=0}^{\infty} \frac{1}{n!} g_n(x,z,t)\, y^n,$$

$$p(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n p}{\partial y^n}\Big|_w y^n \triangleq \sum_{n=0}^{\infty} \frac{1}{n!} d_n(x,z,t)\, y^n.$$

We now seek to express the expansion coefficients $\{a_n, b_n, c_n, d_n, e_n, f_n, g_n\}$ as a function of the externally-applied forcing, $\{F_1, F_2, F_3\}$, and the available data on the wall, which includes the boundary conditions on the velocity $\{u_w, v_w, w_w\}$ and the measurements $\{M_1 \triangleq \frac{\partial u}{\partial y}|_w, M_2 \triangleq p|_w, M_3 \triangleq \frac{\partial w}{\partial y}|_w\}$. We will begin by computing the expansion of the velocity and pressure fields; once these are found, the expansion of the vorticity field follows immediately.

We first observe that computing $\partial^n/\partial y^n$ of the continuity equation (2) results in

$$\frac{\partial^n}{\partial y^n}\frac{\partial v}{\partial y} = -\frac{\partial^n}{\partial y^n}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = -\frac{\partial}{\partial x}\frac{\partial^n u}{\partial y^n} - \frac{\partial}{\partial z}\frac{\partial^n w}{\partial y^n}.$$

Thus, $b_{n+1} = -\partial a_n/\partial x - \partial c_n/\partial z$; *i.e.*, higher-order expansion coefficients for v may be expressed as a simple function of lower-order expansion coefficients for u and w. We note also that the zeroth- and first-order expansion coefficients for u and w and the zeroth-order expansion coefficient for v and p are given by the boundary conditions and measurements. We therefore have

$$a_0 = u_w, \qquad a_1 = M_1, \tag{3a}$$

$$b_0 = v_w, \qquad b_1 = -\frac{\partial a_0}{\partial x} - \frac{\partial c_0}{\partial z}, \tag{3b}$$

$$c_0 = w_w, \qquad c_1 = M_3, \tag{3c}$$

$$d_0 = M_2. \tag{3d}$$

The second-order expansion coefficients for u and w and the first-order expansion coefficient for p may be obtained by rearranging the momentum equations (1) in the following form:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\nu}\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} - \nu\Delta_s u - F_1\right], \tag{4a}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{\nu}\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} - \nu\Delta_s w - F_3\right], \tag{4b}$$

$$\frac{\partial p}{\partial y} = \left[-\frac{\partial v}{\partial t} - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - w\frac{\partial v}{\partial z} + \nu\frac{\partial^2 v}{\partial y^2} + \nu\Delta_s v + F_2\right], \tag{4c}$$

where the surface Laplacian is defined such that $\Delta_s \triangleq \partial^2/\partial x^2 + \partial^2/\partial z^2$. Evaluating at the wall, it follows that

$$a_2 = \frac{1}{\nu}\left[\frac{\partial a_0}{\partial t} + a_0\frac{\partial a_0}{\partial x} + b_0 a_1 + c_0\frac{\partial a_0}{\partial z} + \frac{\partial d_0}{\partial x} - \nu\Delta_s a_0 - F_1\big|_w\right], \tag{5a}$$

$$b_2 = -\frac{\partial a_1}{\partial x} - \frac{\partial c_1}{\partial z}, \tag{5b}$$

$$c_2 = \frac{1}{\nu}\left[\frac{\partial c_0}{\partial t} + a_0\frac{\partial c_0}{\partial x} + b_0 c_1 + c_0\frac{\partial c_0}{\partial z} + \frac{\partial d_0}{\partial z} - \nu\Delta_s c_0 - F_3\big|_w\right], \tag{5c}$$

$$d_1 = \left[-\frac{\partial b_0}{\partial t} - a_0\frac{\partial b_0}{\partial x} - b_0 b_1 - c_0\frac{\partial b_0}{\partial z} + \nu b_2 + \nu\Delta_s b_0 + F_2\big|_w\right]. \tag{5d}$$

Note that, to simplify the derivation, d_n is computed after b_{n+1}. We proceed further by taking $\partial/\partial y$ of (4) and appropriately rearranging the resulting expressions:

$$\frac{\partial^3 u}{\partial y^3} = \frac{1}{\nu}\left[\frac{\partial^2 u}{\partial t\,\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial z\,\partial y} + \frac{\partial^2 p}{\partial x\,\partial y} - \nu\Delta_s\frac{\partial u}{\partial y} - \frac{\partial F_1}{\partial y}\right],$$

$$\frac{\partial^3 w}{\partial y^3} = \frac{1}{\nu}\left[\frac{\partial^2 w}{\partial t\,\partial y} + \frac{\partial u}{\partial y}\frac{\partial w}{\partial x} + u\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial y} + v\frac{\partial^2 w}{\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z} + w\frac{\partial^2 w}{\partial z\,\partial y} + \frac{\partial^2 p}{\partial z\,\partial y} - \nu\Delta_s\frac{\partial w}{\partial y} - \frac{\partial F_3}{\partial y}\right],$$

$$\frac{\partial^2 p}{\partial y^2} = \left[-\frac{\partial^2 v}{\partial t\,\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - u\frac{\partial^2 v}{\partial x\,\partial y} - \left(\frac{\partial v}{\partial y}\right)^2 - v\frac{\partial^2 v}{\partial y^2} - \frac{\partial w}{\partial y}\frac{\partial v}{\partial z} - w\frac{\partial^2 v}{\partial z\,\partial y} + \nu\frac{\partial^3 v}{\partial y^3} + \nu\Delta_s\frac{\partial v}{\partial y} + \frac{\partial F_2}{\partial y}\right].$$

Note that the chain rule has been applied to compute $\partial/\partial y$ of the nonlinear terms. Evaluating at the wall, we obtain

$$a_3 = \frac{1}{\nu}\left[\frac{\partial a_1}{\partial t} + a_1\frac{\partial a_0}{\partial x} + a_0\frac{\partial a_1}{\partial x} + b_1 a_1 + b_0 a_2 + c_1\frac{\partial a_0}{\partial z} + c_0\frac{\partial a_1}{\partial z} + \frac{\partial d_1}{\partial x} - \nu\Delta_s a_1 - \frac{\partial F_1}{\partial y}\bigg|_w\right], \tag{6a}$$

$$b_3 = -\frac{\partial a_2}{\partial x} - \frac{\partial c_2}{\partial z}, \tag{6b}$$

$$c_3 = \frac{1}{\nu}\left[\frac{\partial c_1}{\partial t} + a_1\frac{\partial c_0}{\partial x} + a_0\frac{\partial c_1}{\partial x} + b_1 c_1 + b_0 c_2 + c_1\frac{\partial c_0}{\partial z} + c_0\frac{\partial c_1}{\partial z} + \frac{\partial d_1}{\partial z} - \nu\Delta_s c_1 - \frac{\partial F_3}{\partial y}\bigg|_w\right], \tag{6c}$$

$$d_2 = \left[-\frac{\partial b_1}{\partial t} - a_1\frac{\partial b_0}{\partial x} - a_0\frac{\partial b_1}{\partial x} - b_1^2 - b_0 b_2 - c_1\frac{\partial b_0}{\partial z} - c_0\frac{\partial b_1}{\partial z} + \nu b_3 + \nu\Delta_s b_1 + \frac{\partial F_2}{\partial y}\bigg|_w\right]. \tag{6d}$$

Comparing (5) and (6), a pattern begins to emerge. For all higher-order terms in the expansions of u, v, w, and p, a general formula may now be derived. With $n \geq 4$, we proceed further by taking $\partial^{n-2}/\partial y^{n-2}$ of (4) and appropriately rearranging the resulting expressions:

$$\frac{\partial^n u}{\partial y^n} = \frac{1}{\nu}\left[\frac{\partial^{n-1}u}{\partial t\,\partial y^{n-2}} + \sum_{k=0}^{n-2}\binom{n-2}{k}\left(\frac{\partial^{n-2-k}u}{\partial y^{n-2-k}}\frac{\partial^{k+1}u}{\partial x\,\partial y^k} + \frac{\partial^{n-2-k}v}{\partial y^{n-2-k}}\frac{\partial^{k+1}u}{\partial y^{k+1}} + \frac{\partial^{n-2-k}w}{\partial y^{n-2-k}}\frac{\partial^{k+1}u}{\partial z\,\partial y^k}\right)\right.$$
$$\left. + \frac{\partial^{n-1}p}{\partial x\,\partial y^{n-2}} - \nu\Delta_s\frac{\partial^{n-2}u}{\partial y^{n-2}} - \frac{\partial^{n-2}F_1}{\partial y^{n-2}}\right],$$
$$\frac{\partial^n w}{\partial y^n} = \frac{1}{\nu}\left[\frac{\partial^{n-1}w}{\partial t\,\partial y^{n-2}} + \sum_{k=0}^{n-2}\binom{n-2}{k}\left(\frac{\partial^{n-2-k}u}{\partial y^{n-2-k}}\frac{\partial^{k+1}w}{\partial x\,\partial y^k} + \frac{\partial^{n-2-k}v}{\partial y^{n-2-k}}\frac{\partial^{k+1}w}{\partial y^{k+1}} + \frac{\partial^{n-2-k}w}{\partial y^{n-2-k}}\frac{\partial^{k+1}w}{\partial z\,\partial y^k}\right)\right.$$
$$\left. + \frac{\partial^{n-1}p}{\partial z\,\partial y^{n-2}} - \nu\Delta_s\frac{\partial^{n-2}w}{\partial y^{n-2}} - \frac{\partial^{n-2}F_3}{\partial y^{n-2}}\right],$$
$$\frac{\partial^{n-1}p}{\partial y^{n-1}} = \left[-\frac{\partial^{n-1}v}{\partial t\,\partial y^{n-2}} - \sum_{k=0}^{n-2}\binom{n-2}{k}\left(\frac{\partial^{n-2-k}u}{\partial y^{n-2-k}}\frac{\partial^{k+1}v}{\partial x\,\partial y^k} + \frac{\partial^{n-2-k}v}{\partial y^{n-2-k}}\frac{\partial^{k+1}v}{\partial y^{k+1}} + \frac{\partial^{n-2-k}w}{\partial y^{n-2-k}}\frac{\partial^{k+1}v}{\partial z\,\partial y^k}\right)\right.$$
$$\left. + \nu\frac{\partial^n v}{\partial y^n} + \nu\Delta_s\frac{\partial^{n-2}v}{\partial y^{n-2}} + \frac{\partial^{n-2}F_2}{\partial y^{n-2}}\right].$$

Note that the binomial theorem[2] has been applied to compute $\partial^{n-2}/\partial y^{n-2}$ of the nonlinear terms. Evaluating at the wall, we obtain

$$a_n = \frac{1}{\nu}\left[\frac{\partial a_{n-2}}{\partial t} + \sum_{k=0}^{n-2}\binom{n-2}{k}\left(a_{n-2-k}\frac{\partial a_k}{\partial x} + b_{n-2-k}a_{k+1} + c_{n-2-k}\frac{\partial a_k}{\partial z}\right) + \frac{\partial d_{n-2}}{\partial x} - \nu\Delta_s a_{n-2} - \frac{\partial^{n-2}F_1}{\partial y^{n-2}}\bigg|_w\right],$$
$$b_n = -\frac{\partial a_{n-1}}{\partial x} - \frac{\partial c_{n-1}}{\partial z},$$
$$c_n = \frac{1}{\nu}\left[\frac{\partial c_{n-2}}{\partial t} + \sum_{k=0}^{n-2}\binom{n-2}{k}\left(a_{n-2-k}\frac{\partial c_k}{\partial x} + b_{n-2-k}c_{k+1} + c_{n-2-k}\frac{\partial c_k}{\partial z}\right) + \frac{\partial d_{n-2}}{\partial z} - \nu\Delta_s c_{n-2} - \frac{\partial^{n-2}F_3}{\partial y^{n-2}}\bigg|_w\right],$$
$$d_{n-1} = \left[-\frac{\partial b_{n-2}}{\partial t} - \sum_{k=0}^{n-2}\binom{n-2}{k}\left(a_{n-2-k}\frac{\partial b_k}{\partial x} + b_{n-2-k}b_{k+1} + c_{n-2-k}\frac{\partial b_k}{\partial z}\right) + \nu b_n + \nu\Delta_s b_{n-2} + \frac{\partial^{n-2}F_2}{\partial y^{n-2}}\bigg|_w\right].$$

Combining this result with (3), (5), and (6), it is seen that we may determine *all* terms in the Taylor-series expansions for u, v, w, and p from the current values of the wall measurements of $\partial u/\partial y$, $\partial w/\partial y$, and p and the derivatives of these quantities in x, z, and t, together with knowledge of the externally-applied momentum forcing and the velocity boundary conditions.

The Taylor-series expansions for the vorticity field follow directly from the Taylor-series expansions for the velocity field. Noting the definitions

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \qquad \omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \qquad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

inserting the Taylor-series expansions for the velocity and vorticity components, and matching like powers of y, it follows immediately for all n that

$$e_n = c_{n+1} - \frac{\partial b_n}{\partial z}, \qquad f_n = \frac{\partial a_n}{\partial z} - \frac{\partial c_n}{\partial x}, \qquad g_n = \frac{\partial b_n}{\partial x} - a_{n+1}.$$

2.2 The case with homogeneous boundary conditions

The expressions given above simplify greatly if we take $u_w = v_w = w_w = 0$ and $F_1 = P_x(t)$, $F_2 = F_3 = 0$, as in the case of uncontrolled turbulent channel flow. Defining the notation

$$D_s = \frac{\partial M_1}{\partial x} + \frac{\partial M_3}{\partial z}, \qquad \mathcal{L} = \left(\frac{\partial}{\partial t} - \nu\Delta_s\right), \qquad D_d = \frac{\partial M_1}{\partial x} - \frac{\partial M_3}{\partial z}, \qquad R = \frac{\partial M_3}{\partial x} - \frac{\partial M_1}{\partial z},$$

[2] Given two functions $f(y)$ and $g(y)$, the binomial theorem for differentiation yields $(fg)^{(m)} = \sum_{k=0}^{m}\binom{m}{k} f^{(m-k)}g^{(k)}$.

the first four nonzero terms in the expansions for the velocity, vorticity, and pressure can be written as

$$u(y) = yM_1 + \frac{y^2}{2\nu}\left[\frac{\partial M_2}{\partial x} - P_x\right] + \frac{y^3}{6\nu}\left[\mathcal{L}M_1 - \nu\frac{\partial D_s}{\partial x}\right] + \frac{y^4}{24\nu}\left[\mathcal{L}\frac{1}{\nu}\frac{\partial M_2}{\partial x} - \frac{1}{\nu}\dot{P}_x - \Delta_s\frac{\partial M_2}{\partial x} + M_1 D_d + 2M_3\frac{\partial M_1}{\partial z}\right] + O(y^5),$$

$$v(y) = -\frac{y^2}{2\nu}\nu D_s - \frac{y^3}{6\nu}\Delta_s M_2 - \frac{y^4}{24\nu}\left[\mathcal{L}D_s - \nu\Delta_s D_s\right] - \frac{y^5}{120\nu}\left[\mathcal{L}\frac{1}{\nu}\Delta_s M_2 - \Delta_s\Delta_s M_2 + \frac{\partial}{\partial x}(M_1 D_s) + \frac{\partial}{\partial z}(M_3 D_s) - 4\left(\frac{\partial M_1}{\partial x}\frac{\partial M_3}{\partial z} - \frac{\partial M_3}{\partial x}\frac{\partial M_1}{\partial z}\right)\right] + O(y^6),$$

$$w(y) = yM_3 + \frac{y^2}{2\nu}\frac{\partial M_2}{\partial z} + \frac{y^3}{6\nu}\left[\mathcal{L}M_3 - \nu\frac{\partial D_s}{\partial z}\right] + \frac{y^4}{24\nu}\left[\mathcal{L}\frac{1}{\nu}\frac{\partial M_2}{\partial z} - \Delta_s\frac{\partial M_2}{\partial z} - M_3 D_d + 2M_1\frac{\partial M_3}{\partial x}\right] + O(y^5),$$

$$p(y) = M_2 - y\nu D_s - \frac{y^2}{2}\Delta_s M_2 + \frac{y^3}{6}\nu\Delta_s D_s + O(y^4),$$

$$\omega_x(y) = M_3 + \frac{y}{\nu}\frac{\partial M_2}{\partial z} + \frac{y^2}{2\nu}\mathcal{L}M_3 + \frac{y^3}{6\nu}\left[\mathcal{L}\frac{1}{\nu}\frac{\partial M_2}{\partial z} - M_3 D_d + 2M_1\frac{\partial M_3}{\partial x}\right] + O(y^4),$$

$$\omega_y(y) = -yR - \frac{y^3}{6\nu}\mathcal{L}R + \frac{y^4}{24\nu}\left[M_3\Delta_s M_1 - M_1\Delta_s M_3 - \frac{\partial}{\partial x}(M_1 R) - \frac{\partial}{\partial z}(M_3 R)\right] + O(y^5),$$

$$\omega_z(y) = -M_1 - \frac{y}{\nu}\frac{\partial M_2}{\partial x} - \frac{y^2}{2\nu}\mathcal{L}M_1 - \frac{y^3}{6\nu}\left[\mathcal{L}\frac{1}{\nu}\frac{\partial M_2}{\partial x} + M_1 D_d + 2M_3\frac{\partial M_1}{\partial z}\right] + O(y^4).$$

2.3 The importance of pressure measurements

A natural question to ask as this point is "Can the wall pressure $M_2 \triangleq p_w$ appearing in the above formulae actually be computed from the other information available in this problem, namely $\{u_w, v_w, w_w, M_1 \triangleq \frac{\partial u}{\partial y}|_w, M_3 \triangleq \frac{\partial w}{\partial y}|_w, F_1, F_2, F_3\}$, and therefore not be measured?" The answer to this question appears to be "No", though mathematical proof remains an open problem. Via simple combination of the Navier-Stokes and continuity equations, it is possible to write a 2D Poisson equation for the pressure on the wall. However, in the nonlinear case, it does not appear to be possible to write this 2D Poisson equation in such a manner that the right-hand side depends only on the other information available in this problem formulation. Wall pressure therefore appears to be a key flow measurement which is independent of the wall skin-friction measurements M_1 and M_3.

Note that the wall pressure M_2 plays an important role in the higher-order terms in the Taylor-series expansions derived above; without it, these expansions must be truncated at very low order. Thus, the derivation presented above indicates the key role of pressure measurements in the estimation of the state of the turbulent flow system, regardless of the technique actually used to assimilate these measurements into an estimate of the state of the turbulent flow system.

3 Evaluation of truncated taylor series in DNS of turbulent channel flow

We now investigate the range of validity of the Taylor-series expansions computed in §2.2 subject to various levels of truncation. For this purpose, we use a DNS database for an uncontrolled, constant-mass flux turbulent channel flow at $Re_\tau = 180$ using the spectral/spectral/finite-difference code of Bewley, Moin, & Temam (2001) on a $256 \times 129 \times 256$ numerical grid. Using the wall information (*i.e.*, the measurements M_1, M_2, and M_3) to evaluate the coefficients in the expansions listed in §2.2 (truncated after the i'th-order term), we can reconstruct the velocity and vorticity components and the pressure. The quality of the reconstruction (as a function of the level of truncation, i, and the distance from the wall, y) may be characterized by the correlation of the perturbation components of the reconstructed and actual fields, given by

$$Corr_y(\alpha'_{rec}, \alpha'_{act}) = \frac{\int_0^{L_1}\int_0^{L_3} \alpha'_{rec}(y)\alpha'_{act}(y)\,dx\,dz}{\sqrt{\int_0^{L_1}\int_0^{L_3}(\alpha'_{rec}(y))^2 dx\,dz}\sqrt{\int_0^{L_1}\int_0^{L_3}(\alpha'_{act}(y))^2 dx\,dz}}, \tag{7}$$

where α' denotes the perturbation component (with the mean components subtracted off) of any quantity chosen from the set $\{u, v, w, p, \omega_x, \omega_y, \omega_z\}$, and the subscripts *rec* and *act* correspond to the reconstructed and actual fields respectively. The correlations are computed for the perturbation fields to avoid the bias that might be introduced by the mean field. Thus, the statistics at a given distance from the wall are computed by averaging the instantaneous perturbation fields over the streamwise and spanwise directions; upon discretization, this corresponds to averaging over 2^{16} grid points for each datapoint reported. Spatial differentiation of the wall measurements (in the directions x and z) was carried out spectrally, and temporal differentiation was carried out using a second-order central-difference approximation. In Figure 1, we show the dependence of the correlation (7) for all the quantities in the

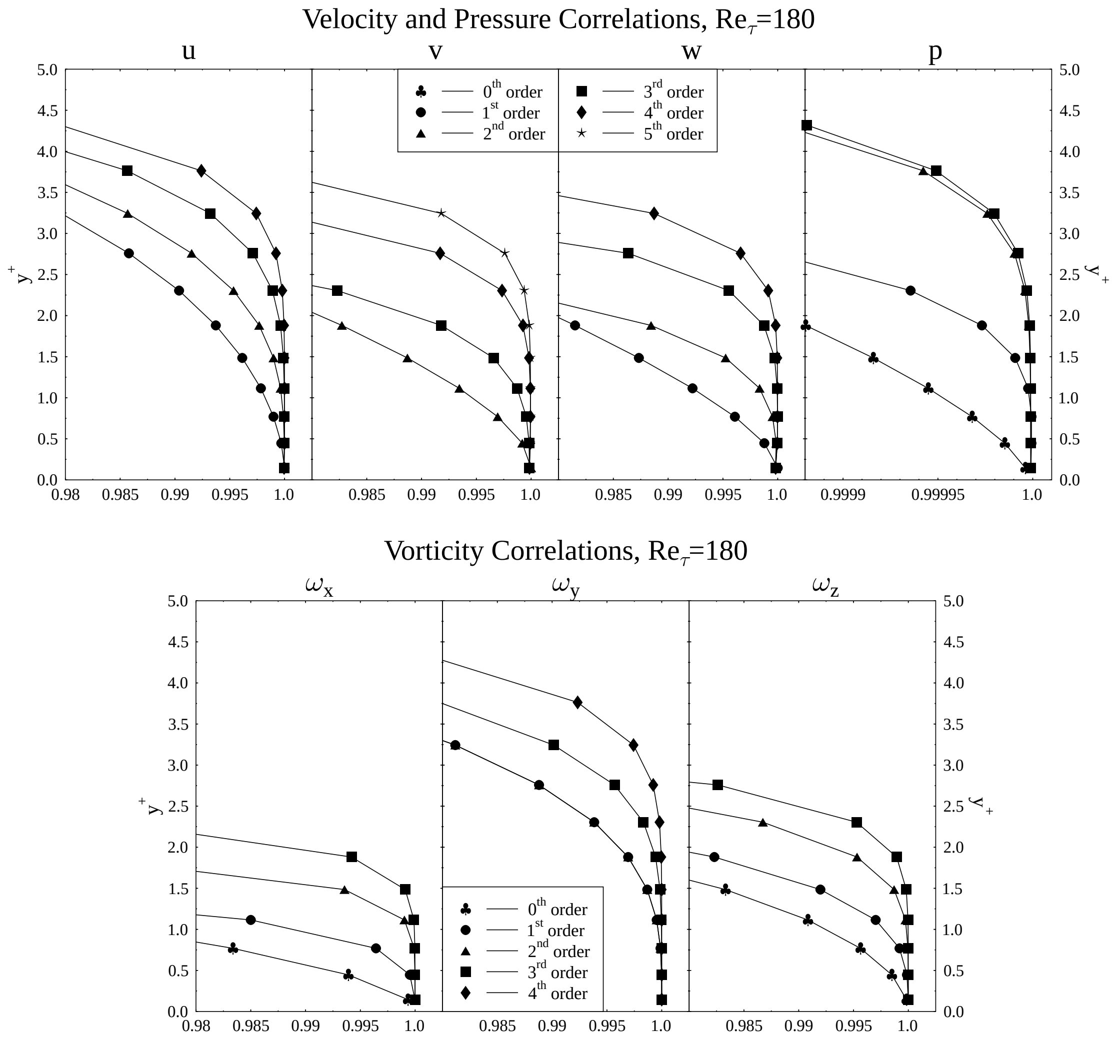

Figure 1: Correlations of the components of the reconstructed and the actual velocity field, pressure field, and vorticity field as a function of the distance from the wall in a turbulent channel flow at $Re_\tau = 180$. Reconstructions were computed by retaining the number of terms indicated in the Taylor-series expansions listed in §2.2, and the correlations were computed according to (7).

set $\{u, v, w, p, \omega_x, \omega_y, \omega_z\}$ as a function of the distance from the wall y and the order of truncation i. The wall–normal coordinate is given in wall units as $y^+ = \frac{y}{\nu/u_\tau}$. In all figures we note a systematic improvement of the reconstruction as more terms are included in expansion. Note that carrying these expansions to even higher orders will eventually be limited by the accuracy of the numerical database.

4 Dynamic state estimation strategies

The above results (in particular, see the comments made in §2.3) highlight the fundamental importance of using all three flow quantities available at the wall when attempting to reconstruct the flow inside the channel in the hypothetical case in which perfect measurements are available on the wall in a neighborhood of time t.

We now make some brief observations concerning the relation of the above findings on the problem of precise state reconstruction with exact measurements to the problem of practical state estimation with noisy measurements in chaotic fluid systems. Such a problem is often referred to as "variational data assimilation" or "4D-var", and plays a central role in the field of numerical weather prediction (for a recent review of this active field of research, see, e.g., Li, Navon, & Zhu 2000). There are essentially two model-based approaches to the problem of state estimation in this setting: adjoint-based strategies and Riccati-based strategies, the latter of which are often based on extended Kalman filters. Complete description of these two approaches is well beyond the scope of the present paper. However, in light of the observations made in the present paper concerning the integral role of wall-pressure measurements in the problem of exact state reconstruction in wall-bounded turbulent flows, it is useful to review the formulation for adjoint-based state estimation in channel-flow systems with noisy measurements at the wall.

Define first an (unknown) noise vector $\mathbf{w} = (w_1\ w_2\ w_3)^T$ and a noisy wall measurement vector $\mathbf{m} = (m_1\ m_2\ m_3)^T$, where $m_1 \triangleq \frac{\partial u_1}{\partial x_2}|_w + w_1$, $m_2 \triangleq p|_w + w_2$, and $m_3 \triangleq \frac{\partial u_3}{\partial x_2}|_w + w_3$, distributed in time over an "assimilation window" $[-T, 0]$ and in space over the channel walls for an "actual" channel-flow system. We now seek to determine the (unknown) initial state Φ of a model system everywhere inside the channel such that, when advanced in time over the interval $-T \to 0$, the model reproduces the observed measurements to the maximum extent possible. We first write the Navier-Stokes equation (1) governing the model system $\mathbf{u} = (u_1\ u_2\ u_3)^T$ in the compact form

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u} + P_x\mathbf{i}, \quad \nabla\cdot\mathbf{u} = 0 && \text{in } \Omega\times(-T,0);\\ &\mathbf{u} = \Phi && \text{at } t=-T;\\ &\mathbf{u}(0,x_2,x_3) = \mathbf{u}(L_1,x_2,x_3), \quad \mathbf{u}(x_1,x_2,0) = \mathbf{u}(x_1,x_2,L_3), \quad \mathbf{u}(x_1,\pm1,x_3) = 0 && \text{on } \partial\Omega. \end{aligned} \tag{8}$$

The objective in the present optimization problem is defined mathematically as the minimization over all feasible initial conditions Φ of a functional $\mathcal{J}(\Phi)$ which represents the "misfit" of the measurements in the actual and reconstructed systems:

$$\mathcal{J}(\Phi) = \frac{1}{2}\int_{-T}^{0}\left[\alpha_1\left\|\frac{\partial u_1}{\partial x_2} - m_1\right\|^2_{\Gamma_2^\pm} + \alpha_2\left\|p - m_2\right\|^2_{\Gamma_2^\pm} + \alpha_3\left\|\frac{\partial u_3}{\partial x_2} - m_3\right\|^2_{\Gamma_2^\pm}\right] dt, \tag{9}$$

where the coefficients α_1, α_2, α_3, and the norm $\|\cdot\|_{\Gamma_2^\pm}$ are defined appropriately to measure the deviation of the model system from the measurements of the actual flow on the channel walls at $x_2 = \pm1$ (denoted here by $\Gamma_2^\pm$). In the present work we will consider the case in which L_2 norms are used such that $\|f\|^2_{\Gamma_2^\pm} \triangleq \int_{\Gamma_2^\pm} f^2\, dS$.

The initial conditions Φ which minimize $\mathcal{J}(\Phi)$ may be found by a gradient-based search using an adjoint-based algorithm. To identify the gradient, an inner product over Ω must first be defined; in the present work, we will consider the L_2 inner product $\langle \mathbf{f}, \mathbf{g}\rangle_\Omega \triangleq \int_\Omega \mathbf{f}\cdot\mathbf{g}\, dV$. The functional gradient $\mathcal{DJ}/\mathcal{D}\Phi$ is then defined such that, for $\varepsilon \ll 1$ and for any Φ',

$$\begin{aligned} \mathcal{J}(\Phi + \varepsilon\Phi') &= \mathcal{J}(\Phi) + \varepsilon\left\langle \frac{\mathcal{DJ}}{\mathcal{D}\Phi}, \Phi'\right\rangle_\Omega = \mathcal{J}(\Phi) + \varepsilon\int_\Omega \frac{\mathcal{DJ}}{\mathcal{D}\Phi}\cdot\Phi'\, dV\\ &= \mathcal{J}(\Phi) + \frac{\varepsilon}{2}\int_{-T}^{0}\int_{\Gamma_2^\pm}\left[\alpha_1\Big(\frac{\partial u_1}{\partial x_2} - m_1\Big)\frac{\partial u_1'}{\partial x_2} + \alpha_2\Big(p - m_2\Big)p' + \alpha_3\Big(\frac{\partial u_3}{\partial x_2} - m_3\Big)\frac{\partial u_3'}{\partial x_2}\right] dS\, dt, \end{aligned} \tag{10}$$

where the equation governing $\mathbf{u}'$ is found by inserting $\Phi + \varepsilon\Phi'$ for Φ and $\mathbf{u} + \varepsilon\mathbf{u}'$ for $\mathbf{u}$ in (8) and collecting the terms proportional to ε; assuming $\varepsilon \ll 1$, this results in

$$\begin{aligned} &\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}' + (\mathbf{u}'\cdot\nabla)\mathbf{u} = -\nabla p' + \nu\Delta\mathbf{u}', \quad \nabla\cdot\mathbf{u}' = 0 && \text{in } \Omega\times(-T,0);\\ &\mathbf{u}' = \Phi' && \text{at } t=-T;\\ &\mathbf{u}'(0,x_2,x_3) = \mathbf{u}'(L_1,x_2,x_3), \quad \mathbf{u}'(x_1,x_2,0) = \mathbf{u}'(x_1,x_2,L_3), \quad \mathbf{u}'(x_1,\pm1,x_3) = 0 && \text{on } \partial\Omega. \end{aligned} \tag{11}$$

Note that (11) reflects a linear relationship between $\mathbf{u}'$ and Φ', though this linear relationship is not yet expressed in a convenient form from which the functional gradient $\mathcal{DJ}/\mathcal{D}\Phi$ may be identified in (10). For this purpose, consider the definition of an "adjoint" state via the equation

$$\begin{aligned} &-\frac{\partial \mathbf{u}^*}{\partial t} + \mathbf{u}\cdot[\nabla\mathbf{u}^* + (\nabla\mathbf{u}^*)^T] = -\nabla p^* + \nu\Delta\mathbf{u}^*, \quad \nabla\cdot\mathbf{u}^* = 0 && \text{in } \Omega\times(-T,0);\\ &\mathbf{u}^* = 0 && \text{at } t=0;\\ &\left.\begin{aligned} &\mathbf{u}^*(0,x_2,x_3) = \mathbf{u}^*(L_1,x_2,x_3), \quad \mathbf{u}^*(x_1,x_2,0) = \mathbf{u}^*(x_1,x_2,L_3),\\ &u_1^*(x_1,\pm1,x_3) = \alpha_1\Big(\frac{\partial u_1}{\partial x_2} - m_1\Big),\\ &u_2^*(x_1,\pm1,x_3) = \alpha_2\Big(p - m_2\Big),\\ &u_3^*(x_1,\pm1,x_3) = \alpha_3\Big(\frac{\partial u_3}{\partial x_2} - m_3\Big), \end{aligned}\right\} && \text{on } \partial\Omega. \end{aligned} \tag{12}$$

Note that, the difficulty involved with numerically solving the adjoint system given above via a backward march from $t = 0$ to $t = -T$ is almost the same as the difficulty involved with solving the original system (8). One slight complication is that the PDE governing $\mathbf{q}^*$ is a function of $\mathbf{q}$, which itself is computed from (8) via a forward march from $t=-T$ to $t=0$. The need for the storage of $\mathbf{q}$ on $[-T,0]$ during this forward march in order to construct the adjoint operator on the backward march can present a significant storage problem. However, this problem is easily averted with a *checkpointing* algorithm which saves $\mathbf{q}$ only occasionally on the forward march, and then recomputes $\mathbf{q}$ as necessary from these "checkpoints" during the backward march for $\mathbf{q}^*$. To see that the functional gradient $\mathcal{DJ}/\mathcal{D}\Phi$ may easily be identified as a simple function of the solution to the adjoint problem defined in (12), define the state vector $\mathbf{q}=\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix}$, the perturbation vector $\mathbf{q}'=\begin{pmatrix}\mathbf{u}'\\ p'\end{pmatrix}$, the adjoint vector $\mathbf{q}^*=\begin{pmatrix}\mathbf{u}^*\\ p^*\end{pmatrix}$, and the linear operators

$$\mathcal{L}\mathbf{q}'=\begin{pmatrix}\dfrac{\partial\mathbf{u}'}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}'+(\mathbf{u}'\cdot\nabla)\mathbf{u}+\nabla p'-\nu\Delta\mathbf{u}'\\ \nabla\cdot\mathbf{u}'\end{pmatrix},\qquad \mathcal{L}^*\mathbf{q}^*=\begin{pmatrix}-\dfrac{\partial\mathbf{u}^*}{\partial t}+\mathbf{u}\cdot[\nabla\mathbf{u}^*+(\nabla\mathbf{u}^*)^T]+\nabla p^*-\nu\Delta\mathbf{u}^*,\\ \nabla\cdot\mathbf{u}^*\end{pmatrix}.$$

The adjoint operator $\mathcal{L}^*$ given above may in fact be determined from the linearized Navier-Stokes operator $\mathcal{L}$ and the L_2 inner product defined by $\langle\mathbf{f},\mathbf{g}\rangle_{\Omega\times(-T,0)}\triangleq\int_{-T}^{0}\int_\Omega\mathbf{f}\cdot\mathbf{g}\,dS\,dt$; straightforward integration by parts (see, e.g., Bewley, Moin, & Temam 2001) leads to an identity of the form

$$\langle\mathbf{q}^*,\mathcal{L}\mathbf{q}'\rangle_{\Omega\times(-T,0)}=\langle\mathcal{L}^*\mathbf{q}^*,\mathbf{q}'\rangle_{\Omega\times(-T,0)}+b, \tag{13}$$

where the operators $\mathcal{L}$ and $\mathcal{L}^*$ are defined above and the boundary terms resulting from the integrations by parts are collected in b:

$$b=\int_\Omega\left(u_j^*u_j'\right)\Big|_{t=0}^{t=T}d\mathbf{x}+\int_0^T\int_{\Gamma_2^\pm}n_j\left[p^*u_j'+u_i^*\left(u_ju_i'+u_j'u_i\right)-\nu\left(u_i^*\frac{\partial u_i'}{\partial x_j}-u_i'\frac{\partial u_i^*}{\partial x_j}\right)+u_j^*p'\right]d\mathbf{x}\,dt.$$

Finally, the identity (13) may be used to put all of the pieces together: inserting the perturbation equation (11) and the adjoint equation (12) into the identity (13) and simplifying, the perturbation of the cost functional given in (10) may be rewritten in the convenient form

$$\int_\Omega\frac{\mathcal{DJ}}{\mathcal{D}\Phi}\cdot\Phi'\,dV=\int_\Omega\mathbf{u}^*|_{t=-T}\cdot\Phi'\,dV.$$

As this derivation is valid for all Φ', we may finally identify the functional gradient which we seek:

$$\frac{\mathcal{DJ}}{\mathcal{D}\Phi}=\mathbf{u}^*|_{t=-T}.$$

The purpose of presenting this derivation in the present paper is to illustrate that there are exactly three possible locations on the boundary for forcing the relevant adjoint equation in this problem, as shown in (12). The misfits of the three measurements m_1, m_2, and m_3 exhaust all possibilities for the forcing of this adjoint problem from the wall. Moreover, given the linearity of the adjoint system with respect to the boundary conditions, the gradient information obtained via the misfits of the three different types of measurements in this problem is linearly additive.

Acknowledgements

This article is dedicated to the memory of Uwe Dallmann, whose characterizations of the "footprints" of separated flows (see, e.g., Theofilis, Hein, & Dallmann 2000) inspired the present investigation.

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