the Hermite kernel function including the quadratic transfer function, that is $G_2$.

(2) If the number of eigenvalues dominating the instantaneous probability densities is finite, this density function can be approximated by the Gram–Charlier expansion. The parameter of this expansion can be estimated from both the linear and quadratic transfer functions.

(3) The consistent results to the quadratic transfer function of response are obtained through cross bispectral analysis in irregular waves and experimental data in amplitude modulation waves and regular waves. As the results it is confirmed that this function has the following characteristics:

- The amplitude parts decrease as the difference frequency of the two wave components becomes high;
- The phase parts do not depend on the sum frequency of the two wave components;
- The diagonal values show the steady drift excursions in regular waves and are not proportional to the squared wave heights.

(4) The transfer function $g$ of horizontal response to slowly instantaneous wave energy, which is introduced newly in this case, is capable to evaluate quantitatively the characteristics of the quadratic transfer function. Within the range of this experiment it is confirmed that the Newman’s approximation can be applied to the quadratic transfer function of external force and that $g$ is nearly equal to the quadratic transfer function of the response.

(5) The linear part of surge response can be separated from the total response in the frequency domain and can be estimated by the usual linear motion prediction method taking into account of the viscous damping. Further it is not affected by the nonlinear part. The response function to external forces at the low frequency motion is different from that obtained from a free oscillation test in still water. As the reason it may be considered that these phenomena are attributed to "increase of damping force in waves" proposed by one of the authors\textsuperscript{25} and Wichers\textsuperscript{26}.

(6) The instantaneous probability distribution of surge responses has the asymmetrical distribution, which broadens towards the direction drifted by waves even though the restoring force is linear. The variance, and the skewness which dominates the asymmetry of the distribution can be estimated from the frequency characteristics of the response.

(7) In order to obtain the instantaneous probability distribution we propose the approximate method matching between the finite Gram–Charlier expansion and the asymptotic form derived from the exact probability density function. The estimated results due to the present method show fairly good agreement with the experimental results.
(8) The new prediction methods for the probability distributions of extremal values and the extreme value are proposed under the assumptions that the response displacement and velocity are independent mutually and the response velocity is of Gaussian distribution with zero mean in addition to the Powell's assumptions. As a result it is confirmed that the Longuet-Higgin's method significantly underestimates the experimental result while the present method is in good agreement with the experimental one.

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**Appendix A**

Exact Solutions of the Instantaneous Probability Density Function of Second Order Response due to the Kac–Siegent Method

From Eq. (2.1.5) the horizontal response of a moored floating structure can be represented as

\[ x(t) = \int_{\tau_1}^{\tau} d\tau_1 g_1(\tau_1) \xi(t - \tau_1) + \int_{\tau_2}^{\tau} d\tau_2 \int_{\tau_1}^{\tau_2} d\tau_1 g_2(\tau_1, \tau_2) \xi(t - \tau_1) \xi(t - \tau_2) \]

\[ = x^{(1)} + x^{(2)} \]  \hfill (A–1)

Let \( \xi(t) \) be an equivalent filtered white noise process, or

\[ \xi(t) = \int_{\tau} d\tau h(\tau) N(t - \tau), \]  \hfill (A–2)

where \( h(\tau) \) is the weighting function and \( N(t) \) is a unit white noise which satisfies

\[ E \{ N(t) N(t - \tau) \} = \delta(\tau), \]  \hfill (A–3)

\( \delta(\tau) \) is the Dirac delta function.

Then following Kac and Siegent, we expand the white noise process in a stochastic series of orthogonal functions as

\[ N(t - \tau) = \sum_{i=1}^{\infty} X_i(t) \phi_i(\tau) \]  \hfill (A–4)

with the normalization

\[ \int_{-\infty}^{\infty} dt \phi_i(t) \phi_j(t) = \delta_{ij}, \]  \hfill (A–5)

where \( X_i(t) \) are the standard Gaussian variables with zero mean and unit variance and they are mutually independent. Then, in terms of the series expansion, the first term in (A–1) becomes

\[ x^{(1)}(t) = \sum_{i=1}^{\infty} c_i X_i(t), \]  \hfill (A–6)

with
\[ c_i = \int_{\sigma} \int_{\tau} g_\delta(\tau) h(\sigma) \phi_i(\tau + \sigma) \, d\tau d\sigma, \quad (A-7) \]

and
\[ x^{(2)}(t) = \sum_{i,j} X_i(t) X_j(t) \int_\alpha \int_\beta \phi_i(\alpha) \phi_j(\beta) S(\alpha,\beta) \, d\alpha d\beta, \quad (A-8) \]

with
\[ S(\alpha,\beta) = \int_\tau \int_\tau h(\alpha - \tau_1) h(\beta - \tau_2) g_\delta(\tau_1,\tau_2) \, d\tau_1 d\tau_2, \quad (A-9) \]

If \( \phi_i \) are chosen as the orthogonal functions which satisfies
\[ \int_\beta S(\alpha,\beta) \phi_j(\beta) \, d\beta = \lambda_j \phi_j(\alpha), \quad (A-10) \]

Eq. (A-8) becomes
\[ x^{(2)} = \sum_{i=1}^\infty \lambda_i X_i(t)^2. \quad (A-11) \]

Thus, Eq. (A-1) is given in the following form:
\[ x(t) = \sum_{i=1}^\infty c_i X_i(t) + \sum_{i=1}^\infty \lambda_i X_i^2(t) \quad (A-12) \]

with
\[ E \{ X_i(t) X_j(t) \} = \delta_{ij} \quad (A-13) \]

The instantaneous probability density function \( p_x \) of \( x(t) \) can be obtained from the inverse Fourier transform of its characteristics function.

The characteristics function is defined by
\[ \phi(\theta) = E \{ \exp(i\theta x) \} = \prod_{j=1}^\infty E \{ \exp(i\theta(c_j X_j + \lambda_j X_j^2)) \}. \quad (A-14) \]

Since \( X_j \) have the probability density function as
\[ p_{x_j}(x) = 1/\sqrt{2\pi} \exp(-x^2/2), \quad (A-15) \]

by using the following identity:
\[ \int_{-\infty}^{\infty} dx \exp(itx - ax^2/2) = \sqrt{2\pi/a} \exp(-t^2/2a) \text{ for } a > 0 \quad (A-16) \]

the characteristics function can be rewritten as
\[ \phi(\theta) = \prod_{j=1}^\infty (1 - 2i\lambda_j \theta)^{-1/2} \exp(-c_j^2 \theta^2/2 (1 - 2i\lambda_j \theta)). \quad (A-17) \]

By the inverse Fourier transform of the characteristics function the instantaneous probability density function of \( x(t) \) becomes
\[ p_x(x) = 1/2\pi \int_{-\infty}^{\infty} d\theta \phi(\theta) \exp(-i\theta x). \quad (A-18) \]

Next we shall consider the integral equation (A-10).

It can be simplified by defining

(362)
\[ \psi_n(t) = \int_u h(u-t) \phi_n(u) du. \] (A-19)

Then Eq. (A-10) can be rewritten as
\[ \int_u duH(t, u) \psi_n(u) = \lambda_n \psi_n(t) \] (A-20)

with
\[ H(t, u) = \int_\tau d\tau g_2(\tau, u) R_\tau(t-\tau), \] (A-21)

where \( R_\tau \) is the auto correlation and is given by
\[ R_\tau(\tau) = \int_{\tau_1} d\tau_1 h(\tau_1) h(\tau + \tau_1). \] (A-22)

Automatically, the normalization relation for the eigenfunctions is obtained as
\[ \int_{\tau_1} d\tau_1 \int_{\tau_2} d\tau_2 \psi_m(\tau_1) \psi_n(\tau_2) g_2(\tau_1, \tau_2) = \lambda_n \delta_{mn}, \] (A-23)

and the parameter \( c_n \) in (A-7) becomes
\[ c_n = \int_\tau d\tau g_1(\tau) \psi_n(\tau). \] (A-24)

Finally we represent the integral equation (A-23) in the frequency domain. If the Fourier transform of \( \psi_n(t) \) exists and is defined by
\[ \Psi_n(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \exp(-i\omega t) \psi_n(t), \] (A-25)

then the integral equation in the time domain, (A-20), can be represented in the following form:
\[ \int_\nu d\nu \Psi_n(\nu) S_\nu(\omega, -\nu) = \lambda_n \Psi_n(\omega). \] (A-26)

The normalization condition (A-24) becomes
\[ \int_{\omega_1} d\omega_1 \int_{\omega_2} d\omega_2 \psi_m(\omega_1) \psi_n(\omega_2) G_2(-\omega_1, -\omega_2) = \lambda_n \delta_{mn}. \] (A-27)

Since Eq. (A-26) is a Fredholm integral equation of the second kind with a Hermite–symmetric kernel, \( \lambda \) will be real and the eigenfunction satisfies
\[ \Psi_n(-\omega) = \Psi_n(\omega)^*, \] (A-28)

where \( * \) denotes the complex conjugate.

**Appendix B**

**Statistical Values of \( x(t) \)**

From Appendix A the horizontal motion of a moored floating structure in irregular waves can be represented as
\[ x = \sum_{i=1}^{\infty} (c_i + \lambda_i X_i) X_i, \quad (B-1) \]

or

\[ x = \int_{\tau_1} d\tau_2 g_1(\tau) \xi(t-\tau) + \int_{\tau_1} d\tau_1 \int_{\tau_2} d\tau_2 g_2(\tau_1, \tau_2) \xi(t-\tau_1) \xi(t-\tau_2). \quad (B-2) \]

From (B-1) the expected values up to third order are given as follows:

\[ E\{x\} = \sum_{i} \lambda_i E\{X_i\} + \sum_{i \neq j} c_i E\{X_i X_j\}, \quad (B-3) \]

\[ E\{x^2\} = \sum_{i,j} c_i c_j E\{X_i X_j\} + \sum_{i,j} \lambda_i \lambda_j E\{X_i X_j^2\}, \quad (B-4) \]

\[ E\{x^3\} = \sum_{i,j,k} c_i c_j c_k E\{X_i X_j X_k\} + \sum_{i,j,k} \lambda_i \lambda_j \lambda_k E\{X_i X_j X_k\} \]
\[ + \sum_{i,j,k} \lambda_i \lambda_j c_k E\{X_i X_j X_k\} + \sum_{i,j,k} \lambda_i c_j \lambda_k E\{X_i X_j X_k\} \]
\[ + \sum_{i,j,k} \lambda_i \lambda_j c_k E\{X_i X_j X_k\} + \sum_{i,j,k} \lambda_i c_j \lambda_k E\{X_i X_j X_k\} \]
\[ + \sum_{i,j,k} \lambda_i \lambda_j \lambda_k E\{X_i X_j X_k\}. \quad (B-5) \]

Since \( X_i (i=1, \ldots, \infty) \) are the standard Gaussian variables with mutual independence, the following relations are satisfied:

\[ E\{X_i\} = 0, \quad (B-6) \]

\[ E\{X_i X_j\} = \delta_{ij}, \quad (B-7) \]

\[ E\{X_i X_j X_k\} = 0, \quad (B-8) \]

\[ E\{X_i X_j X_k X_l\} = \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}, \quad (B-9) \]

\[ E\{X_i X_j X_k X_m X_n\} = 0, \quad (B-10) \]

\[ E\{X_i X_j X_k X_m X_n X_q\} = \delta_{ij} \delta_{kl} \delta_{mn} + \delta_{ij} \delta_{km} \delta_{ln} + \delta_{ij} \delta_{kn} \delta_{lm} + \delta_{ik} \delta_{jl} \delta_{mn} \]
\[ + \delta_{ik} \delta_{jm} \delta_{ln} + \delta_{ik} \delta_{jn} \delta_{lm} + \delta_{ij} \delta_{km} \delta_{ln} + \delta_{ij} \delta_{kn} \delta_{lm} \]
\[ + \delta_{il} \delta_{jm} \delta_{kn} + \delta_{im} \delta_{jk} \delta_{ln} + \delta_{im} \delta_{jn} \delta_{kl} + \delta_{in} \delta_{jm} \delta_{kl} \]
\[ + \delta_{in} \delta_{jm} \delta_{kl}. \quad (B-11) \]

where \( \delta_{ij} \) is the Kronecker delta.

Using the above relations, the mean value \( \bar{x} \), the variance \( \sigma_x^2 \) and the skewness \( \mu \) of \( x(t) \) are obtained as

\[ \bar{x} = E\{x\} = \sum_{i} \lambda_i, \quad (B-12) \]

\[ \sigma_x^2 = E\{x^2\} - \bar{x}^2 = \sum_{i} c_i^2 + 2 \sum_{i} \lambda_i^2, \quad (B-13) \]

\[ \mu \sigma_x^2 = E\{x^3\} - 3E\{x^2\} \cdot \bar{x} + 2\bar{x}^3 = 8 \sum_{i} \lambda_i^3 + 6 \sum_{i} c_i^2 \lambda_i. \quad (B-14) \]
From (B-2) the expected values are written as follows:

\[
E \{x\} = \int_{\tau_1} d\tau_1 \int_{\tau_2} d\tau_2 g_2(\tau_1, \tau_2) R_2(\tau_2 - \tau_1), \tag{B-15}
\]

\[
E \{x^3\} = \int_{\tau_1} d\tau_1 \int_{\tau_2} d\tau_2 \int_{\tau_3} d\tau_3 g_1(\tau_1, \tau_2) g_1(\tau_2, \tau_3) R_3(\tau_3 - \tau_2) + \int_{\tau_1} d\tau_1 \int_{\tau_3} d\tau_3 g_2(\tau_1, \tau_2) \times g_2(\tau_2, \tau_3) \{ R_3(\tau_3 - \tau_2) R_3(\tau_3 - \tau_1) + R_3(\tau_3 - \tau_2) R_3(\tau_3 - \tau_1) \}
\]

\[
E \{x^3\} = \int_{\tau_1} d\tau_1 \cdots \int_{\tau_3} d\tau_3 g_3(\tau_1, \tau_2) g_3(\tau_2, \tau_3) g_3(\tau_3, \tau_4) \{ 6R_3(\tau_3 - \tau_2) R_3(\tau_3 - \tau_1) + 3R_3(\tau_3 - \tau_2) R_3(\tau_3 - \tau_1) \}^3.
\]

Accordingly we can obtain the statistical values of \(x(t)\) as

\[
x = \int_\omega d\omega G_2(\omega, -\omega) S_2(\omega), \tag{B-18}
\]

\[
\sigma_x^2 = \int_\omega d\omega |G_1(\omega)|^2 S_2(\omega) + 2 \int_\omega d\omega_1 \int_\omega d\omega_2 |G_2(\omega_1, \omega_2)|^2 \times S_2(\omega_1) S_2(\omega_2), \tag{B-19}
\]

\[
\mu \sigma_x = \int_\omega d\omega_1 \int_\omega d\omega_2 G_1(-\omega_1) G_1(-\omega_2) G_2(\omega_1, \omega_2) S_2(\omega_1) S_2(\omega_2)
\]

\[
+ 8 \int_\omega d\omega_1 \int_\omega d\omega_2 \int_\omega d\omega_3 \int_\omega d\omega_4 G_2(\omega_1, \omega_2) G_2(\omega_3, \omega_4) G_2(\omega_5, \omega_6) G_2(\omega_7, \omega_8) \times S_2(\omega_1) S_2(\omega_2) S_2(\omega_3) S_2(\omega_4), \tag{B-20}
\]

where \(*\) denotes the complex conjugate.

**Appendix C**

Integral Evaluation to Instantaneous Probability Density Function of \(x(t)\) as \(x \to \infty\)

From Eq. (2.4.2) the instantaneous probability density function of \(x(t)\) may be rewritten as

\[
p_x(x) = \lim_{R \to \infty} 1/2\pi \int_{-R}^{R} \exp(-it\theta) \phi(\theta) d\theta. \tag{C-1}
\]

First, we expand Eq. (C-1) to complex space. By regarding this integral as a complex integral and replacing \(-it\) by \(s\) we obtain the following form:

\[
p_x(x) = \lim_{R \to \infty} 1/2\pi i \int_{-R}^{R} \prod_{j=1} (1 + 2\lambda_j s)^{-1/2} \exp \{ sx + c_j \} ds. \tag{C-2}
\]

Then without loss of generality it can be assumed that \(\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0\) and \(0 > \lambda_{n+1} > \ldots > \lambda_m\).

(365)
For $x \to -\infty$ only the positive eigenvalues are of interest. Thus in the case in which $n$ is even we take the branch cuts from $-1/2\lambda_{2j-1}$ to $-1/2\lambda_{2j}$ $(j=1, \ldots, n/2)$ along the real axis as shown in Fig. (C-1). If $n$ is odd the branch-cut from $-1/2\lambda_s$ to $-\infty$ is added.

![Contours of integration](image)

**Fig. C-1** Contours of integration

Since $1+2\lambda_s$ takes either positive value or negative value when $s$ moves on the negative real axis, we define the branch of $1/\sqrt{1+2\lambda_s}$ as

\[
\begin{align*}
|1+2\lambda_s|^{-1/2} & \text{ for } 1+2\lambda_s > 0, \\
-i|1+2\lambda_s|^{-1/2} & \text{ for } 1+2\lambda_s < 0.
\end{align*}
\]  

(C-3) \hspace{1cm} (C-4)

Then, the integrand of (C-1) is regular except for branch points $-1/2\lambda_j$ $(j=1, \ldots, n)$.

Thus, by Cauchy's theorem the integral along $ABC \sum_j \Gamma_j E$ becomes zero. The integrals along each path are as follows:

1) Integral along $BC$

Setting $s = \text{Re} e^{i\theta} (\pi/2 \leq \theta \leq \pi)$ we have the following inequalities:

\[
|\exp(c_j s^2/2(1+2\lambda_s))| \leq \exp(c_j^2 R \cos \theta/4\lambda_j) \leq 1.
\]

(366)
\[ \left| \exp (sx) \right| = \exp (xR \cos \theta) \leq 1, \]
\[ |1 + 2\lambda_j s|^{-1/2} \leq |1 - 2\lambda_j R|^{-1/2}. \]

Thus the integral along BC becomes zero as \( R \to \infty \).

(2) Integral along EA

It can be proved that the integral along EA also becomes zero as \( R \to \infty \).

(3) Integral along the semi circle of \( \Gamma_j \)

Setting \( s = \rho e^{i\theta} - 1/2\lambda_j (\pi \leq \theta \leq 0) \) we get the following inequalities:

\[ \left| \exp \left( \frac{c_j^2 s^2}{2(1 + 2\lambda_j s)} \right) \right| \leq \exp \left\{ \frac{c_j^2}{2\lambda_j \rho} \left( \rho^2 \cos \theta - \rho / \lambda_j + \cos \theta / \lambda_j \right) \right\}, \]
\[ \left| \exp (sx) \right| \leq \exp (\rho x \cos \theta) \]
\[ |1 + 2\lambda_j s|^{-1/2} \leq 1 / \sqrt{2\lambda_j \rho}. \]

Thus, if \( \rho \) is taken such that it is equal to \( c_j / (2\lambda_j \sqrt{2\lambda_j x}) \), which is a stationary point, this integral becomes zero as \( x \to \infty \).

Since the integrand of (C–1) is bounded at all points except for branch points, we obtain the following form as the evaluation of Eq. (C–1):

\[ p_x(x) = 1/\pi \sum_j (-1)^{j-1} \int_{1/2\lambda_j}^{-1/2\lambda_j} e^{sx} \prod_j |1 + 2\lambda_j s|^{-1/2} \times \exp \left\{ \frac{c_j^2 s^2}{2(1 + 2\lambda_j s)} \right\} \; ds \quad \text{if } n \text{ is even,} \quad \text{(C–5)} \]

\[ p_x(x) = 1/\pi \sum_j (-1)^{j-1} \int_{1/2\lambda_j}^{-1/2\lambda_j} e^{sx} \prod_j |1 + 2\lambda_j s|^{-1/2} \times \exp \left\{ \frac{c_j^2 s^2}{2(1 + 2\lambda_j s)} \right\} \; ds + 1/\pi (-1)^n \int_{-\infty}^{-1/2\lambda_j} e^{sx} \]
\[ \times \prod_j |1 + 2\lambda_j s|^{-1/2} \exp \left\{ \frac{c_j^2 s^2}{2(1 + 2\lambda_j s)} \right\} \; ds \quad \text{if } n \text{ is odd} \quad \text{(C–6)} \]

If it is assumed that \( \lambda_j / \lambda_1 (j = 2, \ldots, n) < 1 \), the main contribution to the integral (C–1) will be from the vicinity of the branch point \( \lambda_1 \).

Similarly the integral as \( x \to -\infty \) is also evaluated by taking the branch-cuts on positive real axis.

In the case of \( n = 2 \) and \( \lambda_1 \lambda_2 < 0 \) and \( \lambda_j > c_j (j = 1, 2) \) the integral (C–1) is evaluated exactly as

\[ p_x(x) = 1/2\pi \int_{-\infty}^{\infty} d\theta \exp (-ix\theta) / \sqrt{(1-2i\lambda_1 \theta)(1-2i\lambda_2 \theta)} \]
\[ = 1/2\pi \sqrt{\lambda_1 \lambda_2} \exp \{ (\lambda_1 - |\lambda_2|) / 4\lambda_1 |\lambda_2| \} K_0 \{ |x|(|\lambda_1 + |\lambda_2|) / 4\lambda_1 |\lambda_2| \}, \quad \text{(C–7)} \]

where \( K_0 \) is the modified Bessel function of the second kind.