# Computation of the Two-Dimensional Incompressible Navier-Stokes Equations for Flow Past a Circular Cylinder Using an Implicit Factored Method*

By

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## ABSTRACTS

An implicit factored method (IFM hereafter for brevity) was used for numerically solving the two-dimensional incompressible Navier-Stokes equations for flow past a circular cylinder at Reynolds numbers of 10, 20, 40, 80, and 160. The pseudo-compressibility was introduced into the continuity equation in order that the IFM can be applied to the equations.

At Re = 10, 20, 40, and 80, steady-state solutions were obtained by iterating in the time domain. The solutions thus obtained were symmetrical with respect to the line of symmetry of the body, and agree well with experimental data. Truncation error analysis was made, and the accuracy of the differences approximating the derivatives in the governing equations was checked. The result showed that the present numerical solutions approximate the real solution with good accuracy, and that the truncation errors are sufficiently small.

At Re = 160, the steady-state solution was not reached, and the flow became unstable and unsymmetrical. The vortex shedding which is similar to that in the real phenomena was observed, though the present scheme is not time-accurate.

Finally, it was concluded that the present scheme is accurate and efficient in solving numerically the incompressible Navier-Stokes equations.

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## 1. INTRODUCTION

An Implicit Factored Method is a finite difference scheme originally developed for numerically solving compressible Navier-Stokes equations. It was founded by Beam and Warming 1), 2) and extended to arbitrary grid geometries by Steger 3).

The characteristics of the method are;

(1) Physical variables such as velocity and pressure are used as dependent variables, so that extension to 3D and inclusion of turbulence models are easy.

(2) Since dependent variables are in vector form, it is suitable for high-speed computation using vector processors.

(3) Use of body fitted coordinates makes the scheme flexible, and application of boundary conditions to bodies of complex geometry becomes straightforward.

(4) Factorization of spatial differencing operators greatly reduces CPU time, and makes computation of large dimensions feasible.

In order to apply the method to a system of partial differential equations, the presence of a time derivative of each dependent variable is necessary. Therefore, a time derivative of pressure is artificially added to the continuity equation, thus introducing "pseudo-compressibility" to the incompressible Navier-Stokes equations 4),14),15). This makes the system hyperbolic, and application of the implicit factored method becomes possible. Non-conservation form is used in spatial differencings. Conservation form is dominantly used with compressible Navier-Stokes equations. The main reason for that is that the conservation form has shock-capturing property in case of transonic and supersonic flow, because it inherently satisfies the Rankine-Hugoniot jump relation. However, in incompressible flows, no shock wave arises, and, as shown in Chapter 6, the non-conservation form has better numerical stability property than the conservation form.

## 2. GOVERNING EQUATIONS

Non-dimensional form of the two-dimensional incompressible Navier-Stokes equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2-1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2-2}$$

$$\frac{\partial p}{\partial t} + \beta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad (2\text{-}3)$$

where $Re \equiv \dfrac{U_\infty L}{\nu}$

, where non-dimensionalization is made using $U_\infty$, freestream speed, $L$, representative length of a body, and $\rho$, the density of the fluid. That is,

$$\begin{cases} u \equiv \dfrac{u^*}{U_\infty}, \ v \equiv \dfrac{v^*}{U_\infty}, \ p \equiv \dfrac{P^*}{\rho U^2_\infty} \\ t \equiv \dfrac{t^*}{L/U_\infty}, x \equiv \dfrac{x^*}{L}, \ y \equiv \dfrac{y^*}{L} \end{cases} \qquad (2\text{-}4)$$

, where * denotes dimensional value. In the following computations, the diameter d of a circular cylinder has been chosen as $L$.

The first term of eq.(2-3) is artificially added to the original continuity equation for incompressible flow, in order to make the system hyperbolic. The addition of the term makes the fluid compressible, thus introducing "pseudo-compressibility".

$\beta$ in the equation is a positive constant. In case of computing a steady-state flow by interating in the time domain, the pseudo-compressibility introduces no error in the converged solution, where all the $\partial/\partial t$ terms vanish, including the added $\partial p/\partial t$ term. Use of large value in $\beta$ allows time-accurate solution, but it makes the system of equations stiff 4).

The above system of equations are written in vector form as shown below.

$$q_t + Fq_x + Gq_y = C_R\,(q_{xx} + q_{yy}) \qquad (2\text{-}5)$$

where

$$q \equiv \begin{bmatrix} u \\ v \\ p \end{bmatrix}, \ F \equiv \begin{bmatrix} u & o & 1 \\ o & u & o \\ \beta & o & o \end{bmatrix}, \ G \equiv \begin{bmatrix} v & o & o \\ o & v & 1 \\ o & \beta & o \end{bmatrix}, \ C_R \equiv \begin{bmatrix} \dfrac{1}{Re} & o & o \\ o & \dfrac{1}{Re} & o \\ o & o & o \end{bmatrix} \qquad (2\text{-}6)$$

The eq.(2-5) is in non-conservation form. The advantage of non-conservation form will be described in detail in Chapter 5.

## 3. COORDINATE TRANSFORMATION

In order to compute a flow around a body of arbitrary shape, it is convenient to use body-fitted coordinates through coodinate transformation in the governing equations. It makes application of boundary condition easy and straightforward, thus making computational scheme simple and flexible.

The coordinate transformation is defined by,

$$\begin{cases} \xi = \xi(x, y) \\ \eta = \eta(x, y) \\ t = t \end{cases} \tag{3-1}$$

, where $(\xi, \eta)$ denotes computational plane and $(x, y)$ denotes the original physical plane.

Partial differenciation operators in $(x, y)$ plane are replaced by those in $(\xi, \eta)$ plane.

$$\begin{cases} \partial_x = \xi_x \partial_\xi + \eta_x \partial_\eta \\ \partial_y = \xi_y \partial_\xi + \eta_y \partial_\eta \end{cases} \tag{3-2}$$

, where,

$$\begin{aligned} &\xi_x = Jy_\eta \ , \ \xi_y = -Jx_\eta \\ &\eta_x = -Jy_\xi \ , \ \eta_y = Jx_\xi \\ &J \equiv \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \frac{1}{\begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix}} \end{aligned} \tag{3-3}$$

$J$ is the Jacobian. Thus, combining eqs. (3-2) and (3-3),

$$\begin{cases} \partial_x = a\partial_\xi + b\partial_\eta \\ \partial_y = c\partial_\xi + d\partial_\eta \end{cases} \tag{3-4}$$

where

$$a \equiv Jy_\eta \ , \quad b \equiv -Jy_\xi, \quad c \equiv -Jx_\eta \ , \quad d \equiv Jx_\xi \tag{3-5}$$

Further, second-order differenciations are, by repeatedly using eq. (3-4),

$$\begin{aligned} \partial_{xx} &= \partial_x(\partial_x) = (a\partial_\xi + b\partial_\eta)(a\partial_\xi + b\partial_\eta) \\ &= a^2\partial_{\xi\xi} + 2ab\partial_{\xi\eta} + b^2\partial_{\eta\eta} + (aa_\xi + ba_\eta)\partial_\xi + (ab_\xi + bb_\eta)\partial_\eta \end{aligned} \tag{3-6}$$

$$\partial_{yy} = c^2\partial_{\xi\xi} + 2cd\partial_{\xi\eta} + d^2\partial_{\xi\eta} + (cc_\xi + dc_\eta)\partial_\xi + (cd_\xi + dd_\eta)\partial_\eta \tag{3-7}$$

Notice that the above relations are in non-conservation form.

The governing equations (2-5) are transformed using eqs. (3-4) through (3-7). The final form is,

$$q_t + Aq_\xi + Bq_\eta = C_R(\hat{a}q_{\xi\xi} + \hat{b}q_{\xi\xi} + \hat{c}q_{\eta\eta} + \hat{d}q_\xi + \hat{e}q_\eta) \tag{3-8}$$

where

$$\begin{cases} A \equiv aF + cG \\ B \equiv bF + dG \end{cases} \tag{3-9}$$

$$\begin{cases} \hat{a} \equiv a^2 + c^2 \\ \hat{b} \equiv 2(ab + cd) \\ \hat{c} \equiv b^2 + d^2 \\ \hat{d} \equiv aa_\xi + ba_\eta + cc_\xi + dc_\eta \\ \hat{e} \equiv ab_\xi + bb_\eta + cd_\xi + dd_\eta \end{cases} \tag{3-10}$$

$$\begin{cases} a_\xi = J_\xi y_\eta + Jy_{\xi\eta} \\ b_\xi = -(J_\xi y_\xi + Jy_{\xi\xi}) \\ c_\xi = -(J_\xi x_\eta + Jx_{\xi\eta}) \\ d_\xi = J_\xi x_\xi + Jx_{\xi\xi} \end{cases} \qquad \begin{cases} a_\eta = J_\eta y_\eta + Jy_{\eta\eta} \\ b_\eta = -(J_\eta y_\xi + Jy_{\xi\eta}) \\ c_\eta = -(J_\eta x_\eta + Jx_{\eta\eta}) \\ d_\eta = J_\eta x_\xi + Jx_{\xi\eta} \end{cases} \tag{3-11}$$

$$\begin{cases} J = \dfrac{1}{S} \\ J_\xi = -J^2 S_\xi \\ J_\eta = -J^2 S_\eta \end{cases} \qquad \begin{cases} S \equiv x_\xi y_\eta - x_\eta y_\xi \\ S_\xi = x_{\xi\xi} y_\eta + x_\xi y_{\xi\eta} - (x_{\xi\eta} y_\xi + x_\eta y_{\xi\xi}) \\ S_\eta = x_{\xi\eta} y_\eta + x_\xi y_{\eta\eta} - (x_{\eta\eta} y_\xi + x_\eta y_{\xi\eta}) \end{cases} \tag{3-12}$$

In actual computation, the equations are discretized in $(\xi, \eta)$ plane. $\xi$- and $\eta$-derivatives in the equations are replaced by differences. Increments in $\xi$- and $\eta$-directions are set to be constant and chosen as unity. At each node in $(\xi, \eta)$ plane, the values of x and y are given, thus defining body and grid geometry, and the values of the geometrical parameters given in eqs. (3-10) through (3-12) are calculated.

## 4. APPROXIMATE FACTORIZATION

**[Padé time differencing]**

A time derivative in eq. (3-8) is replaced by a Padé time differencing.[2)]

$$\frac{\partial q}{\partial t} = \frac{1}{\Delta t} \cdot \frac{\Delta}{1 + \theta\Delta} q^n + \mathrm{O}[(\theta - \frac{1}{2})\Delta t, \Delta t^2] \tag{4-1}$$

where $q^n$ : $q$ at timestep $n$.

$\Delta$ : difference operator.

$$\Delta q^n \equiv q^{n+1} - q^n \tag{4-2}$$

$\theta$ : parameter

$\theta = 0$ ; Euler explicit.

$\theta = 1/2$ ; Trapezoidal.

$\theta = 1$ ; Euler implicit.

The differencing operator eq. (4-1) is second-order accurate when $\theta = 0.5$, and first-order accurate otherwise.

Substituting eq. (4-1) into eq. (3-8),

$$\Delta q^n + \theta\Delta t[\Delta(Aq_\xi)^n + \Delta(Bq_\eta)^n$$
$$- C_R(\hat{a}\Delta q^n{}_{\xi\xi} + \hat{b}\Delta q^n{}_{\xi\eta} + \hat{c}\Delta q^n{}_{\eta\eta} + \hat{d}\Delta q^n_\xi + \hat{e}\Delta q^n{}_\eta)]$$
$$= -\Delta t[A^n q^n{}_\xi + B^n q^n_\eta - C_R(\hat{a}q^n{}_{\xi\xi} + \hat{b}q^n_{\xi\xi} + \hat{c}q^n_{\eta\eta} + \hat{d}q^n_\xi + \hat{e}q^n_\eta)]$$
$$+ O[(\theta - \frac{1}{2})\Delta t^2, \Delta t^3] \tag{4-3}$$

**[Local linearization]**

Nonlinear tems in the above equation are processed using the concept of "local linearization", as follows.

$$\Delta(Aq_\xi) \doteqdot \frac{\partial}{\partial t}(Aq_\xi)\Delta t = \frac{\partial A}{\partial t}\Delta t q_\xi + A\frac{\partial}{\partial t}(q_\xi)\Delta t$$
$$= \frac{\partial A}{\partial t}\Delta t q_\xi + A\frac{\partial}{\partial \xi}(\frac{\partial q}{\partial t}\Delta t) \doteqdot \Delta A q_\xi + A\Delta q_\xi$$
$$= \hat{A}\Delta q + A\Delta q_\xi \tag{4-4}$$

where

$$\hat{A} \equiv \begin{bmatrix} au_\xi & cu_\xi & o \\ av_\xi & cv_\xi & o \\ o & o & o \end{bmatrix} \tag{4-5}$$

Similarly,

$$\Delta(Bq_\eta) \doteqdot \hat{B}\Delta q + B\Delta q_\eta \tag{4-6}$$

where

$$\hat{B} \equiv \begin{bmatrix} bu_\eta & du_\eta & o \\ bv_\eta & dv_\eta & o \\ o & o & o \end{bmatrix} \tag{4-7}$$

Substituting eqs. (4-4) and (4-6) into eq. (4-3), and setting $h \equiv \theta\Delta t$,

$$\Big\{I + h[\hat{A}^n + A^n\frac{\partial}{\partial_\xi} - C_R\Big(\hat{a}\frac{\partial^2}{\partial_\xi{}^2} + \hat{d}\frac{\partial}{\partial_\xi}\Big)]$$
$$+ h[\hat{B}^n + B^n\frac{\partial}{\partial_\eta} - C_R\Big(\hat{c}\frac{\partial^2}{\partial_\eta{}^2} + \hat{e}\frac{\partial}{\partial_\eta}\Big)]\Big\}\Delta q^n \tag{4-8}$$
$$= -\Delta t[Aq_\xi + Bq_\eta - C_R(\hat{a}q_{\xi\xi} + \hat{b}q_{a\eta} + \hat{c}q_{\eta\eta} + \hat{d}q_\xi + \hat{e}q_\eta)]^n + h\hat{b}C_R\Delta q^n_{\xi\eta}$$

**[Explicit treatment of a mixed derivative]**

The mixed derivative in RHS of the above equation is explicitly treated by shifting the timestep from n to n-1.

$$\Delta q_{\xi\eta}^{n} = \Delta q_{\xi\eta}^{n-1} + 0\,[\Delta t^2] \tag{4-9}$$

The above treatment introduces an error of $O(\Delta t^2)$, and does not degrade the solution accuracy.

[Approximate factorization]

The spatial defferenciation operators in LHS of eq. (4-8) are factored as follows. Using eq. (4-9) at the same time,

$$\left\{ I + h[\hat{A} + A\frac{\partial}{\partial\xi} - C_R\left(\hat{a}\frac{\partial^2}{\partial\xi^2} + \hat{d}\frac{\partial}{\partial\xi}\right)] \right\}$$

$$\times \left\{ I + h[\hat{B} + B\frac{\partial}{\partial\eta}\; C_R\left(\hat{c}\frac{\partial^2}{\partial\eta^2} + \hat{e}\frac{\partial}{\partial\eta}\right)] \right\} \Delta q^{n}$$

$$= -\Delta t[Aq_\xi + Bq_\eta - C_R(\hat{a}q_{\xi\xi} + \hat{b}q_{\xi\eta} + \hat{c}q_{\eta\eta} + \hat{d}q_\xi + \hat{e}q_\eta]^n + h\hat{b}\Delta q_{\xi\eta}^{n-1} \tag{4-10}$$

This factorization introduces an error of $O(\Delta t^2)$, thus does not degrade the formal solution accuracy.

By defining an intermediate variable $\Delta q^*$,

$$\Delta q^* \equiv \left\{ I + h\,[\hat{B} + B\frac{\partial}{\partial\eta} - C_R\,(\hat{c}\frac{\partial^2}{\partial\eta^2} + \hat{e}\frac{\partial}{\partial\eta})] \right\} \Delta q^n \tag{4-11}$$

eq. (4-10) is decomposed into two sets ODEs (ordinary differential equations). That is,

$\xi$-sweep

$$\left\{ I + h\;[\hat{A} + A\frac{\partial}{\partial\xi} - C_R\,(\hat{a}\frac{\partial^2}{\partial\xi^2} + \hat{d}\frac{\partial}{\partial\xi})] \right\} \Delta q^*$$

$$= -\Delta t[Aq_\xi + Bq_\eta - C_R\,(\hat{a}q_{\xi\xi} + \hat{b}q_{\xi\eta} + \hat{c}q_{\eta\eta} + \hat{d}q_\xi + \hat{e}q_\eta)]^n$$

$$+ h\hat{b}C_R\,\Delta q_{\xi\eta}^{n-1} - \frac{1}{16}(\omega_\xi\frac{\partial^4}{\partial\xi^4} + \omega_\eta\frac{\partial^4}{\partial\eta^4})q^n \tag{4-12}$$

The last two terms in the above equation are added 4-th order numerical dissipation terms. In general, the addition is necessary at high Reynolds numbers in order to damp numerical disturbances of short wavelength.

$\eta$-sweep

By definition,

$$\left\{ I + h[\hat{B} + B\frac{\partial}{\partial\eta}\; C_R\,(\hat{c}\frac{\partial^2}{\partial\eta^2} + \hat{e}\frac{\partial}{\partial\eta})] \right\} \Delta q^n = \Delta q^* \tag{4-13}$$

Eqs. (4-12) are solved in $\xi$-direction together with appropriate boundary conditions, and then, eqs. (4-13) are solved in $\eta$-direction. This factorization greatly reduces computational work from that of solving an unfactored 2-D boundary value problem.

After solving $\xi$- and $\eta$-sweeps, the values of q at next timestep is,

$$q^{n+1} = q^n + \Delta q^n \tag{4-14}$$

**[Spatial differencings]**

To solve eqs. (4-12) and (4-13), spatial derivatives are approximated by central differencings. Setting $\Delta\xi = \Delta\eta = 1$,

$$\left\{\begin{aligned} \frac{\partial}{\partial\xi} &= \frac{1}{2}(E_\xi^{+1} - E_\xi^{-1}) + O[\Delta\xi^2] \\ \frac{\partial^2}{\partial\xi^2} &= E_\xi^{+1} - 2E^0 + E_\xi^{-1} + O[\Delta\xi^2] \\ \frac{\partial^4}{\partial\xi^4} &= E_\xi^{+2} - 4E_\xi^{+1} + 6E^0 - 4E_\xi^{-1} + E_\xi^{-2} + O[\Delta\xi^2] \end{aligned}\right. \tag{4-15}$$

where $E_\xi^m$ : shifting operator.

$$E_\xi^m q_{i,j} = q_{i+m,j} \quad \begin{array}{l}(i\text{: numbering in } \xi\text{-direction})\\ (j\text{: numbering in } \eta\text{-direction})\end{array}$$

Eta-differencings are defined similarly.

**[Matrix coefficients]**

$\xi$-sweep

Substituting eq. (4-15) into eq. (4-12),

$$L_{ij}\Delta \overset{*}{q}_{i-1,j} + M_{ij}\Delta \overset{*}{q}_{ij} + N_{ij}\Delta \overset{*}{q}_{i+1,j} = f_{\xi ij} \tag{4-16}$$

where

$$\left\{\begin{aligned} L_{ij} &= -h[\frac{1}{2}A + (\hat{a} - \frac{\hat{d}}{2})C_R]_{ij} \\ M_{ij} &= I + h(\hat{A} + 2\hat{a}C_R)_{ij} \\ N_{ij} &= h[\frac{1}{2}A - (\hat{a} + \frac{\hat{d}}{2})C_R]_{ij} \\ f_{\xi ij} &= [RHS \text{ of eq. (4-12)}]_{ij} \end{aligned}\right. \tag{4-17}$$

$\eta$-sweep

$$L_{ij}\Delta q_{i,j-1} + M_{ij}\Delta q_{ij} + N_{ij}\Delta q_{i,j+1} = f_{\eta ij} \tag{4-18}$$

where

$$\begin{cases} L_{ij} = h[\frac{1}{2}B + (\hat{c} - \frac{\hat{e}}{2})C_R]_{ij} \\ M_{ij} = I + h(\hat{B} + 2\hat{c}C_R)_{ij} \\ N_{ij} = h[\frac{1}{2}B - (\hat{c} + \frac{\hat{e}}{2})C_R]_{ij} \\ f_{\eta\, ij} = \Delta q_{ij}^{*} \end{cases} \tag{4-19}$$

Eq. (4-16) or eq. (4-18) forms a block tridiagonal system in general, and is solved efficiently using the Thomas algorithm.

## 5. VON NEUMANN STABILITY ANALYSIS

In this chapter, it will be shown that the non-conservation differencing form used in the present scheme possesses good stability property, using a model scalar equation.

A model scalar equation used is,

$$\frac{\partial U}{\partial t} = -L_{(x,y)}U \tag{5-1}$$

where

$$\begin{cases} L_{(x,y)} \equiv F\frac{\partial}{\partial y} + G\frac{\partial}{\partial y} - R\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \\ F,\ G,\ R \text{ are constants}. \qquad R > 0 \end{cases} \tag{5-2}$$

(1) Stability in (x, y) plane.

Using Padé time differencing (4-1) in eq. (5-1),

$$[1 + \theta\Delta t L_{(x,y)}]\Delta U = -\Delta t L_{(x,y)}U \tag{5-3}$$

Spatial differencings for $x$-derivatives are central differencings similar to eq. (4-15).

$$\begin{cases} \frac{\partial}{\partial x} = \frac{1}{2\Delta x}(E_x^{+1} - E_x^{-1}) \\ \frac{\partial^2}{\partial x^2} = \frac{1}{(\Delta x)^2}(E_x^{+1} - 2E^0 + E_x^{-1}) \end{cases} \tag{5-4}$$

where $E_x^m\, U(x, y) \equiv U(x + m\Delta x, y)$

$Y$-differencings are defined similarly.

An assumed form of solution $U$ is, according to the von Neumann's method[16),

$$U^n = U_0(x, y) + u^n \tag{5-5}$$

where

$$\begin{cases} u^n \equiv v^n\, exp[i(\kappa_1 j\Delta x + \kappa_2 k\Delta y)] \\ v^n = \zeta v^{n-1} = \zeta^2 v^{n-2} = \dots \end{cases} \tag{5-6}$$

$L_{(x,y)} U_0 = 0$

$U^n$: value of $U$ at timestep $n$

$U_0$: steady state solution

$u^n$: assumed periodic disturbance

$\zeta$: amplification factor per each timestep

*RHS* of eq. (5-3) becomes, using eqs. (5-4) through (5-6),

$$[RHS \text{ of eq. (5-3)}] = -\Delta t L_{(x,y)} U^n = -(Re + iIm)u^n \tag{5-7}$$

where

$$\begin{cases} Re \equiv \Delta t R(\kappa_1^2 \alpha^2 + \kappa_2^2 \beta^2) > 0 \\ I_m \equiv \Delta t(F\kappa_1 \alpha \cos\theta_x + G\kappa_2 \beta \cos\theta_y) \\ \alpha \equiv \dfrac{\sin\theta_x}{\theta_x} \quad , \quad \beta \equiv \dfrac{\sin\theta_y}{\theta_y} \\ \theta_x \equiv \dfrac{\kappa_1 \Delta x}{2} \quad , \quad \theta_y \equiv \dfrac{\kappa_2 \Delta y}{2} \end{cases} \tag{5-8}$$

The amplification factor $\zeta$ becomes, substituting eq. (5-7) into eq. (5-3),

$$\zeta = \frac{1 - (1 - \theta)(Re + iIm)}{1 + \theta(Re + iIm)} \tag{5-9}$$

If $|\zeta| < 1$, the scheme is stable and the periodic disturbance diminishes. If $|\zeta| = 1$, it is neutrally stable. If $|\zeta| > 1$, it is unstable and the periodic disturbance grows unboundedly.

From eq. (5-9), the condition $|\zeta| < 1$ leads to the following condition for $\theta$.

$$\theta > \frac{1}{2} - \frac{Re}{Re^2 + Im^2} \tag{5-10}$$

Using the relation $Re > 0$ shown in eq. (5-8), it may be stated that the condition $|\zeta| < 1$ is satisfied for all possible values of $\Delta t, \kappa_1, \kappa_2, \Delta x$ and $\Delta y$ if

$$\theta \geqq \frac{1}{2}$$

That is, the above scheme is unconditionally stable if $\theta \geqq 1/2$.

(2) Stability in $(\xi, \eta)$ plane

The eq. (3-1) is transformed into $(\xi, \eta)$ plane using coordinate transformation defined by eqs. (3-1) through (3-7).

$$\frac{\partial U}{\partial t} = -L_{(\xi,\eta)}U \tag{5-11}$$

where

$$\begin{cases} L_{(\xi,\eta)} = A\dfrac{\partial}{\partial\xi} + B\dfrac{\partial}{\partial\eta} - R\left(\hat{a}\dfrac{\partial^2}{\partial\xi^2} + \hat{b}\dfrac{\partial^2}{\partial\xi\partial\eta} + \hat{c}\dfrac{\partial^2}{\partial\eta^2} + \hat{d}\dfrac{\partial}{\partial\xi} + \hat{e}\dfrac{\partial}{\partial\eta}\right) \\ A \equiv aF + cG \\ B \equiv bF + dG \end{cases} \tag{5-12}$$

Using Padé time differencing (4-1) in eq. (5-11),

$$[1 + \theta\Delta t L_{(\xi,\eta)}]\Delta U = -\Delta t L_{(\xi,\eta)}U \tag{5-13}$$

By assuming a solution of the form similar to eq. (5-5),

$$[RHS \text{ of eq. } (5\text{-}13)] = -\Delta t L_{(\xi,\eta)}U = -(Re + iIm)u^n \tag{5-14}$$

where

$$\begin{cases} Re \equiv \Delta t R(\hat{a}\kappa_1^2\alpha^2 + \hat{b}\kappa_1\kappa_2\alpha\beta cos\theta_\xi cos\theta_\eta + \hat{c}\kappa_2^2\beta^2) \\ I_m \equiv \Delta t[(A - R\hat{d})\kappa_1\alpha cos\theta_\xi + (B - R\hat{e})\kappa_2\beta cos\theta_\eta] \end{cases} \tag{5-15}$$

Further, using eq. (3-10),

$$\begin{aligned} \frac{Re}{\Delta t R} &= (a^2\kappa_1^2\alpha^2 + 2ab\kappa_1\alpha\kappa_2\beta cos\theta_\xi cos\theta_\eta + b\kappa_2^2\beta^2) \\ &\quad + (c^2\kappa_1^2\alpha^2 + 2cd\kappa_1\alpha\kappa_2\beta cos\theta_\xi cos\theta_\eta + d^2\kappa_2^2\beta^2) \\ &\geqq (a^2\kappa_1^2\alpha^2 - 2|ab\kappa_1\alpha\kappa_2\beta| + b^2\kappa_2^2\beta^2) + (\,.\,.\,.\,.\,.\,) \\ &= (|a\kappa_1\alpha| - |b\kappa_2\beta|)^2 + (|c\kappa_1\alpha| - |d\kappa_2\beta|)^2 \\ &\geqq 0 \end{aligned}$$

Therefore,

$$Re \geqq 0 \tag{5-16}$$

Amplification rate $\zeta$ is again given by eq. (5-9). That is;

The present scheme is unconditionally stable under arbitrary coordinate transformations if $\theta \geqq 1/2$.

In the above analysis, the real part $Re$ in eq. (5-15) has no contribution from convection terms of the original equation eq. (5-1). Therefore the positiveness of $Re$ is assured under arbitrary coordinate transformations, which is the key to the unconditional stability above mentioned. On the other hand, the use of conservation form instead of non-conservation form in spatial differencing in the transformed ($\xi$, $\eta$) plane brings contribution from convection terms into $Re$. Therefore, the positiveness of $Re$ is not assured, and the scheme has poorer stability property in general.

The above analysis does not take into account two factors which are included in actual scheme shown by eqs. (4-12) and (4-13). They are, explicit treatment of a mixed derivative, and approximate factorization. The stability analysis on the above two factors are described in detail in ref. 5). According to the analysis it may be stated that explicit treatment of a mixed derivative restricts the stability range, but approximate factorization recovers most of the stability range lost by explicit treatment.

It should be noticed that $\hat{b}$ in eq. (3-10) coincides with $F$ in eq. (A1-2). Therefore, $\hat{b}$ becomes zero if the grid is orthogonal. In that case, the mixed derivative term becomes zero, therefore its explicit treatment does not affect the stability property or time-accuracy.

(3) Stability with added 4-th order numerical dissipation term

4-th order numerical dissipation terms are added to eq. (5-13).

$$[1 + \theta \Delta t L_{(\xi,\eta)}]\Delta U = -\Delta t L_{(\xi,\eta)} U^n - N_{(\xi,\eta)} U^n \tag{5-17}$$

where

$$N_{(\xi,\eta)} \equiv \frac{1}{16}\left(\omega_\xi \frac{\partial^4}{\partial \xi^4} + \omega_\eta \frac{\partial^4}{\partial \eta^4}\right) \qquad \omega_\xi > 0,\ \omega_\eta > 0 \tag{5-18}$$

$\xi$-derivative is approximated using eq. (4-15), and $\eta$-derivative is approximated similarly.

Substituting into eq. (5-17) an assumed solution of the form similar to eq. (5-5), and setting $\theta = 1$ for simplicity,

$$\zeta = \frac{1 - Rn}{1 + Re + iIm} \tag{5-19}$$

where

$$Re,\ Im : \text{given by eq. (5-15)} \tag{5-20}$$

$$Rn \equiv \frac{1}{16}(\omega_\xi \kappa_1^4 \alpha^4 + \omega_\eta \kappa_2^4 \beta^4)$$

From eq. (5-19),

$$|\zeta|^2 = \frac{(1 - Rn)^2}{(1 + Re)^2 + Im^2} \tag{5-21}$$

In order that $|\zeta| \leqq 1$ for all possible values of $Re(>0)$ and Im, Rn must satisfy the following condition.

$$(1-Rn)^2 \leqq 1 \quad \leftrightarrow \quad 0 \leqq Rn \leqq 2 \tag{5-22}$$

The above condition is satisfied if,

$$0 \leqq \omega_\xi \leqq 1 \;\; \text{and} \;\; 0 \leqq \omega_\eta \leqq 1 \tag{5-23}$$

Therefore, it may be stated that the present scheme with added 4-th order numerical dissipation terms is unconditionally stable if $\theta = 1$ and if eq. (5-23) holds, again under arbitrary coordinate transformations.

## 6. TRUNCATION ERROR ANALYSIS

In this chapter, a method is given for estimating the order of accuracy and truncation errors, once computed steady state solution is given.

(1) Steady state equations

The steady state part of eq. (4-10) gives,

$$\begin{aligned} &Aq_\xi + Bq_\eta - C_{\mathrm{R}}(\hat{a}q_{\xi\xi} + \hat{b}q_{\xi\eta} + \hat{c}q_{\eta\eta} + \hat{d}q_\xi + \hat{e}q_\eta) \\ &+ \frac{1}{16\Delta t}(\omega_\xi q_{\xi\xi\xi\xi} + \omega_\eta q_{\eta\eta\eta\eta}) = 0 \end{aligned} \tag{6-1}$$

Three components of the above equation are given respectively as follows.

<u>$x$-momentum eq.</u>

$$\begin{aligned} &\underset{①}{uu_x} + \underset{②}{vu_y} + \underset{③}{p_x} + \underset{④}{\left(-\frac{1}{Re}u_{xx}\right)} + \underset{⑤}{\left(-\frac{1}{Re}u_{yy}\right)} \\ &+ \underset{⑥}{\frac{\omega_\xi}{16\Delta t}u_{\xi\xi\xi\xi}} + \underset{⑦}{\frac{\omega_\eta}{16\Delta t}u_{\eta\eta\eta\eta}} = 0 \end{aligned} \tag{6-2}$$

<u>$y$-momentum eq.</u>

$$\begin{aligned} &\underset{①}{uv_x} + \underset{②}{vv_y} + \underset{③}{p_y} + \underset{④}{\left(-\frac{1}{Re}v_{xx}\right)} + \underset{⑤}{\left(-\frac{1}{Re}v_{yy}\right)} \\ &\underset{⑥}{\frac{\omega_\xi}{16\Delta t}v_{\xi\xi\xi\xi}} + \underset{⑦}{\frac{\omega_\eta}{16\Delta t}v_{\eta\eta\eta\eta}} = 0 \end{aligned} \tag{6-3}$$

continuity eq.

$$\underset{①}{\beta u_x} + \underset{②}{\beta v_y} + \underset{⑥}{\frac{\omega_\xi}{16\Delta t} p_{\xi\xi\xi\xi}} + \underset{⑦}{\frac{\omega_\eta}{16\Delta t} p_{\eta\eta\eta\eta}} = 0 \tag{6-4}$$

**(2) General form of differences and truncation errors**

A derivative of a certain spatial function $f$ is expressed as a sum of its difference and truncation error. Denoting difference as * and truncation error as ~,

$$f_\xi = f_\xi^* + \tilde{f}_\xi \quad \text{where } f_\xi^* \equiv \frac{1}{2}(E_\xi^{+1} - E_\xi^{-1})f,\ \tilde{f}_\xi \equiv -\frac{1}{6} f_{\xi\xi\xi}^* \tag{6-5}$$

$$f_\eta = f_\eta^* + \tilde{f}_\eta \quad \text{where } f_\eta^* \equiv \frac{1}{2}(E_\eta^{+1} - E_\eta^{-1})f,\ \tilde{f}_\eta \equiv -\frac{1}{6} f_{\eta\eta\eta}^* \tag{6-6}$$

$$f_{\xi\xi} = f_{\xi\xi}^* + \tilde{f}_{\xi\xi} \quad \text{where } f_{\xi\xi}^* \equiv (E_\xi^{+1} - 2E^\circ + E_\xi^{-1})f,\ \tilde{f}_{\xi\xi} \equiv -\frac{1}{12} f_{\xi\xi\xi\xi}^* \tag{6-7}$$

$$f_{\eta\eta} = f_{\eta\eta}^* + \tilde{f}_{\eta\eta} \quad \text{where } f_{\eta\eta}^* \equiv (E_\eta^{+1} - 2E^\circ + E_\eta^{-1})f,\ \tilde{f}_{\eta\eta} \equiv -\frac{1}{12} f_{\eta\eta\eta\eta}^* \tag{6-8}$$

$$f_{\xi\eta} = f_{\xi\eta}^* + \tilde{f}_{\xi\eta} \tag{6-9}$$

where $f_{\xi\eta}^* \equiv \frac{1}{4}(E_\xi^{+1} - E_\xi^{-1})(E_\eta^{+1} - E_\eta^{-1}),\ \tilde{f} = -\frac{1}{6}(f_{\xi\eta\eta\eta}^* + f_{\xi\xi\xi\eta}^*)$

$$f_{\xi\xi\xi}^* \equiv (E_\xi^{+2} - 3E_\xi^{+1} + 3E^\circ - E_\xi^{-1})f \quad (2 \leqq i \leqq \text{IM-2}) \tag{6-10}$$

$$f_{\xi\xi\xi\xi}^* \equiv (E_\xi^{+2} - 4E_\xi^{+1} + 6E^\circ - 4E_\xi^{-1} + E_\xi^{-2})f \quad (3 \leqq i \leqq \text{IM-2}) \tag{6-11}$$

$$f_{\xi\xi\xi\eta}^* \equiv \frac{1}{2}(E_\eta^{+1} - E_\eta^{-1})(E_\xi^{+2} - 3E_\xi^{+1} + 3E^\circ - E_\xi^{-1})f \quad (2 \leqq i \leqq \text{IM-2}) \tag{6-12}$$

Similarly with $f_{\xi\eta\eta\eta}^*$.

**(3) Spatial parameters**

Using eqs. (6-5) through (6-12), differences and truncation errors of spatial parameters are given as follows.

$$\begin{cases} x_\xi = \overset{*}{x}_\xi + \tilde{x}_\xi \\ x_\eta = \overset{*}{x}_\eta + \tilde{x}_\eta \\ x_{\xi\xi} = \overset{*}{x}_{\xi\xi} + \tilde{x}_{\xi\xi} \\ x_{\xi\eta} = \overset{*}{x}_{\xi\eta} + \tilde{x}_{\xi\eta} \\ x_{\eta\eta} = \overset{*}{x}_{\eta\eta} + \tilde{x}_{\eta\eta} \end{cases} \tag{6-13}$$

Similarly with $y$.

Following eqs. (6-14) through (6-21) are given using eqs. (3-10) through (3-12). It is assumed that truncation error is small compared with diffe-erence, and only the first-order terms are picked up.

$$S = S^* + \tilde{S} \quad \text{where} \begin{cases} S^* = \overset{*}{x}_\xi \overset{*}{y}_\eta - \overset{*}{x}_\eta \overset{*}{y}_\xi \\ \tilde{S} = \overset{*}{x}_\xi \tilde{y}_\eta + \tilde{x}_\xi \overset{*}{y}_\eta - \overset{*}{x}_\eta \tilde{y}_\xi - \tilde{x}_\eta \overset{*}{y}_\xi \end{cases} \tag{6-14}$$

$$S_\xi = S_\xi^* + \tilde{S}_\xi \tag{6-15}$$

$$\text{where } \tilde{S}_\xi = \overset{*}{x}_{\xi\xi}\tilde{y}_\eta + \tilde{x}_{\xi\xi}\overset{*}{y}_\eta + \overset{*}{x}_\xi \tilde{y}_{\xi\eta} + \tilde{x}_\xi \overset{*}{y}_{\xi\eta} - (\overset{*}{x}_{\xi\eta}\tilde{y}_\xi + \tilde{x}_{\xi\eta}\overset{*}{y}_\xi + \overset{*}{x}_\eta \tilde{y}_{\xi\xi} + \tilde{x}_\eta \overset{*}{y}_{\xi\xi})$$

$$S_\eta = S_\eta^* + \tilde{S}_\eta \tag{6-16}$$

$$\text{where } \tilde{S}_\eta = \overset{*}{x}_{\xi\eta}\tilde{y}_\eta + \tilde{x}_{\xi\eta}\overset{*}{y}_\eta + \overset{*}{x}_\xi \tilde{y}_{\eta\eta} + \tilde{x}_\xi \overset{*}{y}_{\eta\eta} - (\overset{*}{x}_{\eta\eta}\tilde{y}_\xi + \tilde{x}_{\eta\eta}\overset{*}{y}_\xi + \overset{*}{x}_\eta \tilde{y}_{\xi\eta} + \tilde{x}_\eta \overset{*}{y}_{\xi\eta})$$

$$J = J^* + \tilde{J} \quad \text{where } \tilde{J} = -(J^*)^2 \tilde{S} \tag{6-17}$$

$$J_\xi = J_\xi^* + \tilde{J}_\xi \quad \text{where } \tilde{J}_\xi = -J^*(J^*\tilde{S}_\xi + 2\tilde{J}S_\xi^*) \tag{6-18}$$

$$J_\eta = J_\eta^* + \tilde{J}_\eta \quad \text{where } \tilde{J}_\eta = -J^*(J^*\tilde{S}_\eta + 2\tilde{J}S_\eta^*) \tag{6-19}$$

$$a = a^* + \tilde{a} \quad \text{where } \tilde{a} = J^*\tilde{y}_\eta + \tilde{J}\overset{*}{y}_\eta \tag{6-20}$$

Similarly with $b$, $c$, $d$.

$$a_\xi = a_\xi^* + \tilde{a}_\xi \quad \text{where } \tilde{a}_\xi = J_\xi^*\tilde{y}_\eta + \tilde{J}_\xi \overset{*}{y}_\eta + J^*\tilde{y}_{\xi\eta} + \tilde{J}\overset{*}{y}_{\xi\eta} \tag{6-21}$$

Similarly with $b_\xi$, $c_\xi$, $d_\xi$, $a_\eta$, $b_\eta$, $c_\eta$, and $d_\eta$.

**(4) Differenced form of steady state equation**

Using the results shown in the previous two sections, each term of the components of the steady state equations (6-2) through (6-4) is expressed as follows.

$x$-momentum equation

① $uu_x = [uu_x]^* + \widetilde{uu_x}$ (6-22)

where
$$\begin{cases} [uu_x]^* = u(a^* u_\xi^* + b^* u_\eta^*) \\ \widetilde{uu_x} = u(a^* \tilde{u}_\xi + \tilde{a} u_\xi^* + b^* \tilde{u}_\eta + \tilde{b} u_\eta^*) \end{cases}$$

② $vu_y = [vu_y]^* + \widetilde{vu_y}$ (6-23)

where
$$\begin{cases} [vu_y]^* = v(c^* u_\xi^* + d^* u_\eta^*) \\ \widetilde{vu_y} = v(c^* \tilde{u}_\xi + \tilde{c} u_\xi^* + d^* \tilde{u}_\eta + \tilde{d} u_\eta^*) \end{cases}$$

③ $p_x = p_x^* + \tilde{p}_x$ (6-24)

where
$$\begin{cases} p_x^* = a^* p_\xi^* + b^* p_\eta^* \\ \tilde{p}_x = a^* \tilde{p}_\xi + \tilde{a} p_\xi^* + b^* \tilde{p}_\eta + \tilde{b} p_\eta^* \end{cases}$$

④ $-\frac{1}{Re} u_{xx} = [-\frac{1}{Re} u_{xx}]^* + \widetilde{[-\frac{1}{Re} u_{xx}]}$ (6-25)

where
$$[-\frac{1}{Re} u_{xx}]^* = -\frac{1}{Re} [a^{*2} u_{\xi\xi}^* + 2a^* b^* u_{\xi\eta} + b^{*2} u_{\eta\eta}^*$$
$$+ (a^* a_\xi^* + b^* a_\eta^*) u_\xi^* + (a^* b_\xi^* + b^* b_\xi^*) u_\eta^*]$$
$$\widetilde{[-\frac{1}{Re} u_{xx}]} = -\frac{1}{Re} [2a^* \tilde{a} u_{\xi\xi}^* + a^{*2} \tilde{u}_{\xi\xi}$$
$$+ 2(a^* b^* \tilde{u}_{\xi\eta} + a^* \tilde{b} u_{\xi\eta}^* + \tilde{a} b^* u_{\xi\eta}^*)$$
$$+ 2b^* \tilde{b} u_{\eta\eta}^* + b^{*2} \tilde{u}_{\eta\eta}$$
$$+ (a^* a_\xi^* + b^* a_\eta^*) \tilde{u}_\xi$$
$$+ (a^* \tilde{a}_\xi + \tilde{a} a_\xi^* + b^* \tilde{a}_\eta + \tilde{b} a_\eta^*) u_\xi^*$$
$$+ (a^* b_\xi^* + b^* b_\eta^*) \tilde{u}_\eta$$
$$+ (a^* \tilde{b}_\xi + \tilde{a} b_\xi^* + b^* \tilde{b}_\eta + \tilde{b} b_\eta^*) u_\eta^*]$$

⑤ $-\frac{1}{Re} u_{yy} = [-\frac{1}{Re} u_{yy}]^* + \widetilde{[-\frac{1}{Re} u_{yy}]}$ (6-26)

where $[-\frac{1}{Re} u_{yy}]^*$ and $\widetilde{[-\frac{1}{Re} u_{yy}]}$ are expressed by changing $a$ to $c$ and $b$ to $d$ in eq. (6-25).

$$\textcircled{6}\quad \frac{\omega_\xi}{16\Delta t}\, u_{\xi\xi\xi\xi} = \frac{\omega_\xi}{16\Delta t}\, u_{\xi\xi\xi\xi}{}^{*} \tag{6-27}$$

Estimation of truncation error is not necessary.

$$\textcircled{7}\quad \frac{\omega_\eta}{16\Delta t}\, u_{\eta\eta\eta\eta} = \frac{\omega_\eta}{16\Delta t}\, u_{\eta\eta\eta\eta}{}^{*} \tag{6-28}$$

Similarly with $y$-momentum equation and continuity equation.

**(5) Differences and truncation errors in steady state equations**

Using the results shown in the previous section, derivatives in the steady state equations are approximated by differences, producing truncation errors.

<u>$x$-momentum equation</u>

$$\underset{\textcircled{1}^*}{[uu_x]^*} + \underset{\textcircled{2}^*}{[vu_y]^*} + \underset{\textcircled{3}^*}{p_x^*} + \underset{\textcircled{4}^*}{[-\frac{1}{Re}u_{xx}]^*} + \underset{\textcircled{5}^*}{[-\frac{1}{Re}u_{yy}]^*}$$

$$+ \underset{\textcircled{6}^*}{\frac{\omega_\xi}{16\Delta t}U_{\xi\xi\xi\xi}} + \underset{\textcircled{7}^*}{\frac{\omega_\eta}{16\Delta t}u_{\eta\eta\eta\eta}} = [\text{residual}] \tag{6-29}$$

$$\underset{\widetilde{\textcircled{1}}}{\widetilde{uu_x}} \quad \underset{\widetilde{\textcircled{2}}}{\widetilde{vu_y}} \quad \underset{\widetilde{\textcircled{3}}}{\widetilde{p_x}} \quad \underset{\widetilde{\textcircled{4}}}{\widetilde{[-\frac{1}{Re}u_{xx}]}} \quad \underset{\widetilde{\textcircled{5}}}{\widetilde{[-\frac{1}{Re}u_{yy}]}}$$

<u>$y$-momentum equation</u>

$$\underset{\textcircled{1}^*}{[uv_x]^*} + \underset{\textcircled{2}^*}{[vv_y]^*} + \underset{\textcircled{3}^*}{P_y^*} + \underset{\textcircled{4}^*}{[-\frac{1}{Re}v_{xx}]^*} + \underset{\textcircled{5}^*}{[-\frac{1}{Re}V_{\eta\eta}]^*}$$

$$+ \underset{\textcircled{6}^*}{\frac{\omega_\xi}{16\Delta t}V_{\xi\xi\xi\xi}^*} + \underset{\textcircled{7}^*}{\frac{\omega_\xi}{16\Delta t}v_{\eta\eta\eta\eta}^*} = [\text{residual}] \tag{6-30}$$

$$\underset{\widetilde{\textcircled{1}}}{[\widetilde{uv_x}]} \quad \underset{\widetilde{\textcircled{2}}}{[\widetilde{vv_y}]} \quad \underset{\widetilde{\textcircled{3}}}{\widetilde{P_y}} \quad \underset{\widetilde{\textcircled{4}}}{\widetilde{[-\frac{1}{Re}v_{xx}]}} \quad \underset{\widetilde{\textcircled{5}}}{\widetilde{[-\frac{1}{Re}v_{yy}]}}$$

<u>continuity equation</u>

$$\underset{\textcircled{1}^*}{\beta u_x^*} + \underset{\textcircled{2}^*}{\beta v_y^*} + \underset{\textcircled{6}^*}{\frac{\omega_\xi}{16\Delta t}P_{\xi\xi\xi\xi}^*} + \underset{\textcircled{7}^*}{\frac{\omega_\eta}{16\Delta t}P_{\eta\eta\eta\eta}^*} = [\text{residual}] \tag{6-31}$$

$$\underset{\widetilde{\textcircled{1}}}{\widetilde{\beta u_x}} \quad \underset{\widetilde{\textcircled{2}}}{\widetilde{\beta v_y}}$$

Residuals in the above equations show how closely the given numerical solution reaches a steady state, and truncation errors show how accurately the differences approximate the derivatives. The terms with ~ below the equations show that the truncation error arises in the term above it. The layout of the terms in the above three equations coincides with that in Table 8-1, which will be shown in Chapter 8.

(6) **Use of the same differencing operators in metrics and flow variables**

In this section, it will be shown that;

> No truncation error arises under arbitrary coordinate transformations if the same differencing operators are used in metrics and flow variables, and if the flow variables are linear with $x$ and $y$.

Let us suppose that a flow variable $f$ which represents $u$, $v$, and $p$ is linear with $x$ and $y$.

$$f \equiv \alpha x + \beta y \tag{6-32}$$

where $\alpha$, $\beta$ are constants.

Hereafter, in this section, derivatives are denoted as $\dfrac{\partial f}{\partial x}$, and differences are denoted by subscripts, such as $f_x$.

i) 1st derivative

From eq. (6-32),

$$\frac{\partial f}{\partial x} = \alpha \tag{6-33}$$

$\xi$- and $\eta$-differences of the linear function $f$ defined by eq. (6-23) are expressed as below, under the assumption that the same differencing operators are used in $f$, $x$, and $y$.

$$f_\xi = \alpha x_\xi + \beta y_\xi \quad , \quad f_\eta = \alpha x_\eta + \beta y_\eta \tag{6-34}$$

$X$-difference of $f$ is then expressed, using eqs. (3-3) through (3-5),

$$f_x = \frac{y_\eta(\alpha x_\xi + \beta y_\xi) - y_\xi(\alpha x_\eta + \beta y_\eta)}{x_\xi y_\eta - x_\eta y_\xi} = \alpha \tag{6-35}$$

Therefore,

$$f_x = \frac{\partial f}{\partial x} \tag{6-36}$$

That is, $x$-differencing of $f$ produces no truncation error. Similar results may be obtained with $y$-differencing.

ii) 2nd derivatives

From eq. (6-32),

$$\frac{\partial^2 f}{\partial x^2} = 0 \tag{6-37}$$

2nd difference of $f$ with respect to $x$ is expressed using eq. (3-6),

$$f_{xx} = a^2 f_{\xi\xi} + 2ab f_{\xi\eta} + b^2 f_{\eta\eta} + (aa_\xi + ba_\eta) f_\xi + (ab_\xi + bb_\eta) f_\eta$$

$$\downarrow$$

$$\frac{f_{xx}}{J^2} = y_\eta^2 f_{\xi\xi} - 2y_\xi y_\eta f_{\xi\eta} + y_\xi^2 f_{\eta\eta} + J(S_\eta y_\xi - S_\xi y_\eta)(f_\xi y_\eta - f_\eta y_\xi)$$
$$+ y_{\xi\eta}(f_\xi y_\eta + f_\eta y_\xi) - y_{\eta\eta} y_\xi f_\xi - y_{\xi\xi} y_\eta f_\eta \tag{6-38}$$

2nd difference of $f$ is decomposed into two parts using eq. (6-32), that is,

$$f_{xx} = (\alpha x + \beta y)_{xx} = \alpha x_{xx} + \beta y_{xx} \tag{6-39}$$

Substituting $x$ and $y$ respectively in place of $f$ in eq. (6-38),

$$\frac{x_{xx}}{J^2} = \cdots = 0 \quad \text{and} \quad \frac{y_{xx}}{J^2} = \cdots = 0 \tag{6-40}$$

Therefore,

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 0 \tag{6-41}$$

Similarly with $f_y$, $f_{yy}$, and $f_{xy}$.

The use of the same differencing operators on metrics and flow variables is very important for two reasons. One reason is that it assures that the differenced form of equations approximates the original differential equations more accurately as the mesh becomes finer and the flow variables can be more accurately regarded as linear locally. The other is that the uniform flow away from the solid body is accurately expressed using non-uniform mesh, because all the flow variables are linear (i.e., constant) there.

## 7. BOUNDARY CONDITIONS

### (1) Grid system

The grid system used in the present calculation is an "O-grid". The physical $(x, y)$ plane is mapped onto the computational $(\xi, \eta)$ plane by making a cut along $\eta$-axis, as shown in Fig. 7-1. The surface of the body in question is mapped onto the bottom boundary in $(\xi, \eta)$ plane.

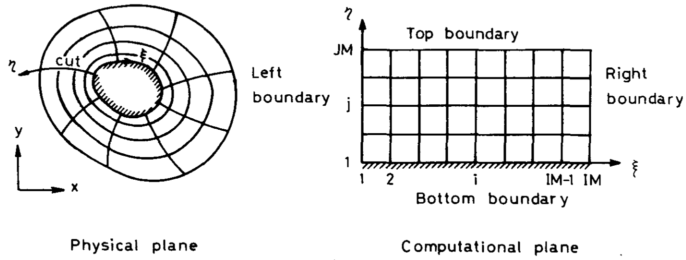


Fig. 7-1 O-grif system.

## (2) $\xi$-sweep

Boundary conditions on left and right boundaries are needed in $\xi$-sweep. There the periodic boundary condition is imposed because the boundaries form a single cut in $(x, y)$ plane. That is,

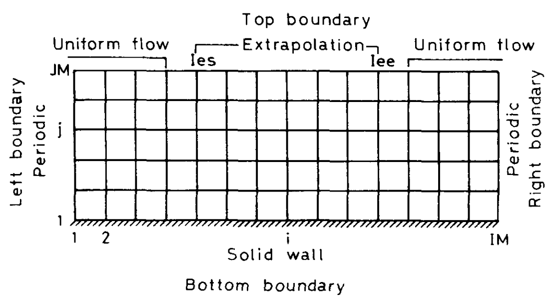


Fig. 7-2 Boundary conditions.

$$x_{1+k,j} = x_{\mathrm{IM}+k,j} \quad , \quad y_{1+k,j} = y_{\mathrm{IM}+k,j} \qquad (7\text{-}1)$$

$$(k = 0, \pm 1, \pm 2, \ldots)$$

$$q_{1+k,j} = q_{\mathrm{IM}+k,j} \quad , \quad \Delta q_{1+k,j} = \Delta q_{\mathrm{IM}+k,j} \qquad (7\text{-}2)$$

$$(k = 0, \pm 1, \pm 2, \ldots)$$

Periodicity of the intermediate variable $\Delta q^*$ is easily shown using eqs. (4-11), (7-1), and (7-2). That is,

$$\Delta q^*_{1+k,j} = \Delta q^*_{\mathrm{IM}+k,j} \qquad (k = 0, \pm 1, \pm 2, \ldots) \qquad (7\text{-}3)$$

Using eq. (7-3) as boundary conditions on left and right boundaries, the matrix coefficients shown in eqs. (4-16) and (4-17) form a block periodic tridiagonal system.

$$\begin{bmatrix} M_1 & N_1 & & & & L_1 \\ L_2 & M_2 & N_2 & & \mathrm{O} & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & O & & L_{IM-2} & M_{IM-2} & N_{IM-2} \\ N_{IM-1} & & & & L_{IM-1} & M_{IM-1} \end{bmatrix} \begin{bmatrix} \Delta q_1^* \\ \Delta q_2^* \\ \cdot \\ \cdot \\ \cdot \\ \Delta q_{IM-2}^* \\ \Delta q_{IM-1}^* \end{bmatrix} = \begin{bmatrix} f_{\xi 1} \\ f_{\xi 2} \\ \cdot \\ \cdot \\ \cdot \\ f_{\xi\, IM-2} \\ f_{\xi\, IM-1} \end{bmatrix} \quad (7\text{-}4)$$

Solution algorithm for block periodic tridiagonal system is available, though it is about twice as costly as that for ordinary tridiagonal system.

(3) $\eta$-sweep

i) Bottom boundary

Solid wall boundary condition is imposed on entire bottom boundary. They are, using eq. (A1-12),

$$\begin{cases} u = 0 \\ v = 0 \\ p_\eta = \dfrac{1}{Re}(\tilde{a}\, u_{\eta\eta} + \tilde{b}\, v_{\eta\eta} + \tilde{c}\, u_\eta + \tilde{d}\, v_\eta) \end{cases} \quad (7\text{-}5)$$

, where $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, and $\tilde{d}$ are defined in eq. (A1-13). $\eta$-differencings on the bottom boundary are expressed as,

$$\begin{cases} \dfrac{\partial}{\partial \eta_{j=1}} = E_\eta^{+1} - E^0 \\ \dfrac{\partial^2}{\partial \eta^2_{j=1}} = E_\eta^{+2} - 2E_\eta^{+1} + E^0 \end{cases} \quad (7\text{-}6)$$

Differenciating the above equation by $t$, and approximating it by difference using eq. (4-15) plus the following formulas on bottom boundary result in

$$\Delta q_1 = a\mathrm{l}^B + B^B \Delta q_2 + C^B \Delta q_3 \quad (7\text{-}7)$$

, where

$$a\mathrm{l}^B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, B^B = \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ \dfrac{2\tilde{a}-\tilde{c}}{Re} & \dfrac{2\tilde{b}-\tilde{d}}{Re} & 1 \end{bmatrix}, \ C^B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\dfrac{\tilde{a}}{Re} & -\dfrac{\tilde{b}}{Re} & 0 \end{bmatrix} \quad (7\text{-}8)$$

ii) Top boundary

The top boundary is a closed loop which surrounds the body with a large radius. The uniform flow boundary condition is imposed in most parts, except for the wake region, where the flow is not uniform and the extrapolation boundary condition is imposed instead. The wake region is defined as $Ies \leqq i \leqq Iee$, where $Ies$ and $Iee$ are properly chosen to cover the wake region.

That is,

Uniform flow ($1 < i \leqq = Ies$-1 or $Iee + 1 < i \leqq$ IM)

$$q_{i,JM} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{7-9}$$

Differenciating the above equation by $t$ and approximating it by diffeences,

$$\Delta q_{JM} = a\mathrm{I}^{T} + B^{T} \Delta q_{JM-1} + C^{T} \Delta q_{JM-2} \tag{7-10}$$

, where

$$a\mathrm{I}^{T} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, B^{T} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, C^{T} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7-11}$$

Extrapolation ($Ies \leqq i \leqq Iee$)

The extrapolation condition is given by,

$$\frac{\partial q}{\partial x} = 0 \tag{7-12}$$

Differencing the above equation by $t$,

$$\frac{\partial \Delta q}{\partial x} = 0 \tag{7-13}$$

Using eq. (3-4) and explicitly treating the $\xi$-differencing,

$$\Delta q_{\eta}^{n} = -\frac{a}{b} \Delta q_{\xi}^{n-1} \tag{7-14}$$

Therefore, eq. (7-10) is again used with the equation below instead of eq. (7-11).

$$a\mathrm{I}^{T} = -\frac{a_{JM}}{b_{JM}} \Delta q_{\xi JM}^{n-1}, \quad B^{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C^{T} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7-15}$$

Eq. (4-18) together with eqs. (7-7) and (7-10) form a block tridiagonal system.

$$\begin{bmatrix} M_2+L_2B^B & N_2+L_2C^B & & & & O \\ L_3 & M_3 & N_3 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ O & & & L_{JM-2} & M_{JM-2} & N_{JM-2} \\ & & & & L_{JM-1}+N_{JM-1}C^T & M_{JM-1}+N_{JM-1}B^T \end{bmatrix} \begin{bmatrix} \Delta q_2 \\ \Delta q_3 \\ \cdot \\ \cdot \\ \cdot \\ \Delta q_{JM-2} \\ \Delta q_{JM-1} \end{bmatrix}$$

$$= \begin{bmatrix} f_{\eta 2}-L_2\,al^B \\ f_{\eta 3} \\ \cdot \\ \cdot \\ \cdot \\ f_{\eta JM-2} \\ f_{\eta JM-1}-N_{JM-1}\,al^T \end{bmatrix} \tag{7-16}$$

4-th order numerical dissipation terms in $\xi$-sweep are differenced near top and bottom boundaries as follows. Near bottom boundary, that is, at $j = 2$,

$$\frac{\partial^4}{\partial \eta^4} f_2 = f_4 - 4f_3 + 6f_2 - 4f_1 + f_0$$

Linear extrapolation is used for giving $f$.

$$f_0 = 2f_1 - f_2$$

Then,

$$\frac{\partial^4}{\partial \eta^4} f_2 = f_4 - 4f_3 + 5f_2 - 2f_1 \tag{7-17}$$

Near top boundary, that is at j=JM-1, using extrapolation $f_{JM+1} = f_{JM}$,

$$\frac{\partial^4}{\partial \eta^4} f_{JM-1} = -3f_{JM} + 6f_{JM-1} - 4f_{JM-2} + f_{JM-3} \tag{7-18}$$

## (4) Initial conditions

Since steady-state solutions are pursued in the present context, initial conditions are arbitrary in principle. However, they must be compatible with the boundary conditions eqs. (7-2), (7-5), (7-9), and (7-12), because an updated $q$, which is the sum of $q$ at a previous timestep and $\Delta q$, must satisfy the given boundary conditions at each timestep. The initial conditions thus chosen are as follows.

i) $u$

$$\begin{cases} u_{ij} = 0 & (1 \leqq i \leqq IM \text{ and } j = 1, 2, 3) \\ u_{ij} = 1 & (1 \leqq i \leqq IM \text{ and } j = JM\text{-}1, JM) \\ u_{ij} = \dfrac{j-3}{JM-4} & (1 \leqq i \leqq IM \text{ and } 4 \leqq j \leqq JM\text{-}2) \end{cases} \tag{7-19}$$

, where, in the last equation, $u$ is linearly interpolated at intermediate values of $j$.

ii) $v$

$$v_{ij} = 0 \qquad (1 \leqq i \leqq IM \text{ and } 1 \leqq j \leqq JM) \tag{7-20}$$

iii) $p$

$$p_{ij} = 0 \qquad (1 \leqq i \leqq IM \text{ and } 1 \leqq j \leqq JM) \tag{7-21}$$

**(5) Updating**

i) q$(i, j)$ $(1 \leqq i \leqq IM\text{-}1$ and $2 \leqq j \leqq JM\text{-}1)$

Given by solving Eqs. (7-4) and (7-16).

ii) $q(i, 1)$ $(1 \leqq i \leqq IM\text{-}1)$

Bottom boundary condition is used, i.e., eq. (7-7).

iii) $q(i, JM)$ $(1 \leqq i \leqq IM\text{-}1)$

In the uniform flow region, the top boundary condition eq. (7-10) is used with eq. (7-11) or (7-13).

In the extrapolation boundary condition region, the updated $\Delta q$ in the inner region is used to determine $\Delta q$ on the top boundary, in order to satisfy exactly the eq. (7-12) at each timestep. That is,

$$\frac{a}{2}(\Delta q_{i+1,JM} - \Delta q_{i-1,JM}) + b(\Delta q_{i,JM} - \Delta q_{i,JM-1}) = 0 \tag{7-22}$$

$$(Ies \leqq i \leqq Iee)$$

The above equations are solved using the tridiagonal solver.

iv) $q(IM, j)$ $(1 \leqq j \leqq JM)$

Periodic boundary condition is used, i.e., eq. (7-2).

After going through i) to iv), eq. (4-14) is used for updating $q$.