the Hermite kernel function including the quadratic transfer function, that is G_2.

(2) If the number of eigenvalues dominating the instantaneous probability densities is finite, this density function can be approximated by the Gram -Charlier expansion. The parameter of this expansion can be estimated from both the linear and quadratic transfer functions.

(3) The consistent results to the quadratic transfer function of response are obtained through cross bispectral analysis in irregular waves and experimental data in amplitude modulation waves and regular waves. As the results it is confirmed that this function has the following characteristics:

-The amplitude parts decrease as the difference frequency of the two wave components becomes high;

-The phase parts do not depend on the sum frequency of the two wave components;

-The diagonal values show the steady drift excursions in regular waves and are not proportional to the squared wave heights.

(4) The transfer function g of horizontal response to slowly instantaneous wave energy, which is introduced newly in this case, is capable to evaluate quantitatively the characteristics of the quadratic transfer function. Within the range of this experiment it is confirmed that the Newman's approximation can be applied to the quadratic transfer function of external force and that g is nearly equal to the quadratic transfer function of the response.

(5) The linear part of surge response can be separated from the total response in the frequency domain and can be estimated by the usual linear motion prediction method taking into account of the viscous damping. Further it is not affected by the nonlinear part. The response function to external forces at the low frequency motion is different from that obtained from a free oscillation test in still water. As the reason it may be considered that these phenomena are attributed to "increase of damping force in waves" proposed by one of the authors[25] and Wichers[26].

(6) The instantaneous probability distribution of surge responses has the asymmetrical distribution, which broadens towards the direction drifted by waves even though the restoring force is linear. The variance, and the skewness which dominates the asymmetry of the distribution can be estimated from the frequency characteristics of the response.

(7) In order to obtain the instantaneous probability distribution we propose the approximate method matching between the finite Gram-Charlier expansion and the asymptotic form derived from the exact probability density function. The estimated results due to the present method show fairly good agreement with the experimental results.

(8) The new prediction methods for the probability distributions of extremal values and the extreme value are proposed under the assumptions that the response displacement and velocity are independent mutually and the response velocity is of Gaussian distribution with zero mean in addition to the Powell's assumptions. As a result it is confirmed that the Longuet-Higgin's method significantly underestimates the experimental result while the present method is in good agreement with the experimental one.

6. ACKNOWLEDGEMENT

This work has been supported in part by the special coordination fund for promoting S&D of the Science and Technology Agency of the Japanese Government, as a part of the Research on the Utilization of Marine Space by Coastal and Offshore Structures.

The authors would like to acknowledge the assistance in the experiments of Messrs, K. Hoshino, who is one of the research staff in Ocean Engineering Division of S. R. I. and Y. Kawakami, who is the Section Head of Ship Propulsion Division and we would like to appreciate Mr. Kagemoto and Dr. Yue for use of the calculation program of hydrodynamic forces acting on the assembly of columns which has been developed by them.

The authors are also grateful to Dr. Takaishi, who is the Director of Ship Dynamics Division, the research staff in Ocean Engineering Division and Prof. Kinoshita of the University of Tokyo for valuable discussion and criticism.

The computations were carried out on the Fujitsu FACOM M-180 IIAD computer at the central computer center of Ship Research Institute.

REFERENCES

1) Hasselmann, K.: On Nonlinear Ship Motions in Irregular Waves, J. S. R., Vol. 10, No. 1 (1966).
2) Dalzel, J. F.: Cross-Bispectral Analysis: Application to Ship Resistance in Waves, J. S. R., Vol. 18, No. 1 (1974).
3) Neal, E.: Second-Order Hydrodynamic Forces due to Stochastic Excitation, Proceedings of 10th ONR (1974).
4) Vinje, T.: On the Calculation of Maxima of Nonlinear Wave Forces and Wave Induced Motion, I. S. P., Vol. 23, No. 268 (1976).
5) Vinje, T.: On the Statistical Distribution of Second Order Forces and Motion, I. S. P., Vol. 30, No. 343 (1983)
6) Hineno, M.: The Calculation of the Statistical Distribution of the Maxima of Nonlinear Response in Irregular Waves, J. S. N. A., Vol. 156 (1984)
7) Takai, R., Ando, S., Kato, S. and Ohkawa, Y.: On the Study of the Mooring Forces Acting on the Multi-Mooring Lines of a Huge-Scale Floating

Platform, Papers of S. R. I., supplement No. 6 (1985).
8) Naess, A.: Statistical Analysis of Second-Order Response of Marine Structure, J. S. R., Vol. 29, No. 4 (1985).
9) Kato, S. and Ando, S.: On the Statistical Prediction of Horizontal Motions of a Moored Floating Structure in Random Waves, J. S. N. A.,Vol. 158 (1985).
10) Kac, M. and Siegert, A. J. F.: On the Theory of Noize in Radio Receiver with Square-Law Detectors, J. A. P., Vol. 18 (1946).
11) Bedrosian, E. and Rice, S. O.: The Output Properties of Volterra Systems (Nonlinear Systems with Memory) Driven by Harmonic and Gaussian Inputs, Proceedings of IEEE, Vol. 59, No. 12 (1971).
12) Volterra, V.: Theory of Functional and Integral and Integro Differential Equations, Blackie and Sons, Ltd., London (1930).
13) Tick, L. J.: The Estimation of Transfer Functions of Quadratic Systems, Technometrics, Vol. 3, No. 4 (1961).
14) Pinkster, J. A.: Low Frequency Phenomena Associated with Vessels Moored at Sea, Society of P. E. AIME, SPE No. 4837 (1974).
15) Huston, W. B. and Skopinski, T. H.: Probability and Frequency Characteristics of Some Flight Buffet Loads, NACA TN3733 (1956).
16) Cartwright, D. E. and Longuet-Higgins, M. S.: The Statistical Distribution of the Maxima of a Random Function, Proceedings of the Royal Society, Vol. 237 (1956).
17) Lin, Y. K.: Probabilistic Theory of Structural Dynamics, Mcgraw-Hill, Newyork, (1967).
18) Kato, S., Ando, S. and Hoshino, K.: On the Maximum Drift Excursions of a Moored Floating Structure, Proceedings of the 46th Ship Research Institute, No. 1 (1985).
19) Cramér, H.: Mathematical Method of Statistics, Princeton University Press (1946).
20) Newman, J. N.: Second-Order, Slowly-Varying Forces on Vessels in Irregular Waves, International Symposium on Marine Vehicle and Structure in Waves, London (1974).
21) Kato, S. and Ando, S.: Estimation of Behaviour of Huge Scale Platform in Waves, Papers of S. R. I., supplement No. 6 (1985).
22) Kagemoto, H. and Yue, D. K. P.: Wave Forces on Multiple Platforms, Proceedings of the 4th International Conference on Behaviour of Offshore Structures (1985).
23) Longuet-Higgins, M. S.: On the Statistical Distribution of the Heights of Sea Waves, J. M. R., No. 11 (1952).
24) Hineno, M.: A Calculation of the Statistical Distribution of the Maxima of

Nonlinear Response in Irregular Waves (2nd Report), J. S. N. A., No. 159 (1986).
25) Kato, S. and Kinoshita, T.: On the Effect of Exciting Short Period Disturbance on Free and Forced Oscillation for the System with Nonlinear Damping, 36th Performance Subcommittee of Ocean Engineering Committee, the Soc. of Nav. Archit. of Japan, (1983).
26) Wichers, J. E. W.: On the Low-Frequency Surge Motions of Vessels Moored in High Seas, O. T. C. paper No. 4437 (1982).

Appendix A

Exact Solutions of the Instantaneous Probability Density Function of Second Order Response due to the Kac-Siegert Method[10)]

From Eq. (2.1.5) the horizontal response of a moored floating structure can be represented as

$$\begin{aligned} \mathrm{x}(t) &= \int_{\tau} d\tau g_1(\tau)\,\xi(t-\tau) + \int_{\tau_1} d\tau_1 \int_{\tau_2} d\tau_2 g_2(\tau_1, \tau_2)\,\xi(t-\tau_1)\,\xi(t-\tau_2) \\ &= \mathrm{x}^{(1)} + \mathrm{x}^{(2)} \end{aligned} \tag{A-1}$$

Let $\xi(t)$ be an equivalent filtered white noize process, or

$$\xi(t) = \int_{\tau} d\tau h(\tau) N(t-\tau), \tag{A-2}$$

where $h(\tau)$ is the weighting function and $N(t)$ is a unit white noize which satisfies

$$E\{N(t)N(t-\tau)\} = \delta(\tau), \tag{A-3}$$

$\delta(\tau)$ is the Dirac delta function.

Then following Kac and Siegert[10)], we expand the white noize process in a stochastic series of orthogonal functions as

$$N(t-\tau) = \sum_{i=1}^{\infty} X_i(t)\,\phi_i(\tau) \tag{A-4}$$

with the normalization

$$\int_{-\infty}^{\infty} dt \phi_i(t)\,\phi_j(t) = \delta_{ij}, \tag{A-5}$$

where $X_j(t)$ are the standard Gaussian variables with zero mean and unit variance and they are mutually independent. Then, in terms of the series expansion, the first term in (A-1) becomes

$$\mathrm{x}^{(1)}(t) = \sum_{i=1}^{\infty} c_i X_i(t), \tag{A-6}$$

with

$$c_i = \int_\sigma \int_\tau g_1(\tau)\, h(\sigma)\, \phi_i(\tau+\sigma)\, d\tau d\sigma, \tag{A-7}$$

and

$$\mathrm{x}^{(2)}(t) = \sum_{i,j} X_i(t)\, X_j(t) \int_\alpha \int_\beta \phi_i(\alpha)\, \phi_j(\beta)\, S(\alpha,\beta)\, d\alpha d\beta, \tag{A-8}$$

with

$$S(\alpha,\beta) = \int_{\tau_1} \int_{\tau_2} h(\alpha-\tau_1)\, h(\beta-\tau_2)\, g_2(\tau_1,\tau_2)\, d\tau_1 d\tau_2. \tag{A-9}$$

If ϕ_i are chosen as the orthogonal functions which satisfies

$$\int_\beta S(\alpha,\beta)\, \phi_j(\beta)\, d\beta = \lambda_j \phi_j(\alpha), \tag{A-10}$$

Eq. (A-8) becomes

$$\mathrm{x}^{(2)} = \sum_{i=1}^{\infty} \lambda_i X_i^2. \tag{A-11}$$

Thus, Eq. (A-1) is given in the following form:

$$\mathrm{x}(t) = \sum_{i=1}^{\infty} c_i X_i(t) + \sum_{i=1}^{\infty} \lambda_i X_i^2(t) \tag{A-12}$$

with

$$E\ \{X_i(t)\, X_j(t)\} = \delta_{ij} \tag{A-13}$$

The instantaneous probability density function p_{x} of $\mathrm{x}(t)$ can be obtained from the inverse Fourier transform of its characteristics function.

The characteristics function is defined by

$$\phi(\theta) = E\{\exp(i\theta \mathrm{x})\} = \prod_{j=1}^{\infty} E\{\exp\{i\theta(c_j X_j + \lambda_j X_j^2)\}\}. \tag{A-14}$$

Since X_j have the probability density function as

$$p_{\mathrm{x}j}(x) = 1/\sqrt{2\pi}\, \exp(-x^2/2), \tag{A-15}$$

by using the following identity:

$$\int_{-\infty}^{\infty} dx \exp(itx - ax^2/2) = \sqrt{2\pi/a}\, \exp(-t^2/2a)\ \ for\ a>0 \tag{A-16}$$

the characteristic function can be rewritten as

$$\phi(\theta) = \prod_{j=1}^{\infty} (1-2i\lambda_j\theta)^{-1/2} \exp\{-c_j^2 \theta^2/2(1-2i\lambda_j\theta)\}. \tag{A-17}$$

By the inverse Fourier transform of the characteristics function the instantaneous probability density function of $\mathrm{x}(t)$ becomes

$$p_{\mathrm{x}}(x) = 1/2\pi \int_{-\infty}^{\infty} d\theta \phi(\theta) \exp(-i\theta x). \tag{A-18}$$

Next we shall consider the integral equation (A-10).

It can be simplified by defining

$$\psi_n(t) = \int_u h(u-t)\,\phi_n(u)\,du. \tag{A-19}$$

Then Eq. (A-10) can be rewritten as

$$\int_u duH(t, u)\,\psi_n(u) = \lambda_n \psi_n(t) \tag{A-20}$$

with

$$H(t, u) = \int_\tau d\tau g_2(\tau, u)\,R_\zeta(t-\tau), \tag{A-21}$$

where R_ζ is the auto correlation and is given by

$$R_\zeta(\tau) = \int_{\tau_1} d\tau_1 h(\tau_1)\,h(\tau+\tau_1). \tag{A-22}$$

Automatically, the normalization relation for the eigenfunctions is obtained as

$$\int_{\tau_1} d\tau_1 \int_{\tau_2} d\tau_2 \psi_m(\tau_1)\,\psi_n(\tau_2)\,g_2(\tau_1, \tau_2) = \lambda_n \delta_{mn}, \tag{A-23}$$

and the parameter c_n in (A-7) becomes

$$c_n = \int_\tau d\tau g_1(\tau)\,\psi_n(\tau). \tag{A-24}$$

Finally we represent the integral equation (A-23) in the frequency domain. If the Fourier transform of $\Psi_n(t)$ exists and is defined by

$$\Psi_n(\omega) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} dt\,\exp(-i\omega t)\,\psi_n(t), \tag{A-25}$$

then the integral equation in the time domain, (A-20), can be represented in the following form:

$$\int_\nu d\nu \Psi_n(\nu)\,S_\zeta(\omega)\,G_2(\omega, -\nu) = \lambda_n \Psi_n(\omega). \tag{A-26}$$

The normalization condition (A-24) becomes

$$\int_{\omega_1} d\omega_1 \int_{\omega_2} d\omega_2 \Psi_n(\omega_1)\,\Psi_m(\omega_2)\,G_2(-\omega_1, -\omega_2) = \lambda_n \delta_{mn}. \tag{A-27}$$

Since Eq. (A-26) is a Fredholm integral equation of the second kind with a Hermite-symmetric kernel, λ will be real and the eigenfunction satisfies

$$\Psi_n(-\omega) = \Psi_n^*(\omega), \tag{A-28}$$

where * denotes the complex conjugate.

Appendix B

Statistical Values of x(t)

From Appendix A the horizontal motion of a moored floating structure in irregular waves can be represented as

$$x=\sum_{i=1}^{\infty}(c_i+\lambda_i X_i)X_i, \tag{B-1}$$

or

$$x=\int_{\tau}d\tau g_1(\tau)\xi(t-\tau)+\int_{\tau_1}d\tau_1\int_{\tau_2}d\tau_2 g_2(\tau_1,\tau_2)\xi(t-\tau_1)\xi(t-\tau_2). \tag{B-2}$$

From (B-1) the expected values up to third order are given as follows;

$$E\{x\}=\sum_i \lambda_i E\{X_i^2\}+\sum_i c_i E\{X_i\}, \tag{B-3}$$

$$E\{x^2\}=\sum_{i,j} c_i c_j E\{X_i X_j\}+\sum_{i,j} c_i\lambda_j E\{X_i X_j^2\}, \tag{B-4}$$

$$\begin{aligned}E\{x^3\}=&\sum_{i,j,k} c_i c_j c_k E\{X_i X_j X_k\}+\sum_{i,j,k} c_i\lambda_j c_k E\{X_i X_j^2 X_k\}\\&+\sum_{i,j,k}\lambda_i c_j c_k E\{X_i^2 X_j X_k\}+\sum_{i,j,k}\lambda_i\lambda_j c_k E\{X_i^2 X_j^2 X_k\}\\&+\sum_{i,j,k} c_i c_j\lambda_k E\{X_i X_j X_k^2\}+\sum_{i,j,k} c_i\lambda_j\lambda_k E\{X_i X_j^2 X_k^2\}\\&+\sum_{i,j,k}\lambda_i c_j\lambda_k E\{X_i^2 X_j X_k^2\}+\sum_{i,j,k}\lambda_i\lambda_j\lambda_k E\{X_i^2 X_j^2 X_k^2\}.\end{aligned} \tag{B-5}$$

Since $X_i(i=1,\ldots,\infty)$ are the standard Gaussian variables with mutual independence, the following relations are satisfied:

$$E\{X_i\}=0, \tag{B-6}$$

$$E\{X_i X_j\}=\delta_{ij}, \tag{B-7}$$

$$E\{X_i X_j X_k\}=0, \tag{B-8}$$

$$E\{X_i X_j X_k X_l\}=\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}, \tag{B-9}$$

$$E\{X_i X_j X_k X_l X_m\}=0, \tag{B-10}$$

$$\begin{aligned}E\{X_i X_j X_k X_l X_m X_n\}=&\delta_{ij}\delta_{kl}\delta_{mn}+\delta_{ij}\delta_{km}\delta_{ln}+\delta_{ij}\delta_{kn}\delta_{lm}+\delta_{ik}\delta_{jl}\delta_{mn}\\&+\delta_{ik}\delta_{jm}\delta_{ln}+\delta_{ik}\delta_{jn}\delta_{lm}+\delta_{il}\delta_{jk}\delta_{mn}+\delta_{il}\delta_{jm}\delta_{kn}\\&+\delta_{il}\delta_{jn}\delta_{km}+\delta_{im}\delta_{jk}\delta_{ln}+\delta_{im}\delta_{jl}\delta_{kn}+\delta_{im}\delta_{jn}\delta_{kl}+\delta_{in}\delta_{jk}\delta_{lm}\\&+\delta_{in}\delta_{jl}\delta_{km}+\delta_{in}\delta_{jm}\delta_{kl},\\&\textit{for } i,j,l,m,n=1,\ldots,\infty,\end{aligned} \tag{B-11}$$

where δ_{ij} is the Kronecker delta.

Using the above relations, the mean value $\bar{x}$, the variance σ_x^2 and the skewness μ of $x(t)$ are obtained as

$$\bar{x}=E\{x\}=\sum_i \lambda_i, \tag{B-12}$$

$$\sigma_x^2=E\{x^2\}-\bar{x}^2=\sum_i c_i^2+2\sum_i \lambda_i^2, \tag{B-13}$$

$$\mu\sigma_x^3=E\{x^3\}-3E\{x^2\}\cdot\bar{x}+2\bar{x}^3=8\sum_i \lambda_i^3+6\sum_i c_i^2\lambda_i. \tag{B-14}$$

From (B−2) the expected values are written as follows:

$$E\{x\}=\int_{\tau_1} d\tau_1 \int_{\tau_2} d\tau_2 g_2(\tau_1, \tau_2) R_\xi(\tau_2-\tau_1), \tag{B−15}$$

$$E\{x^2\}=\int_{\tau_1} d\tau_1 \int_{\tau_2} d\tau_2 g_1(\tau_1) g_1(\tau_2) R_\xi(\tau_2-\tau_1)+\int_{\tau_1} d\tau_1 \cdots\cdots \int_{\tau_4} d\tau_4 g_2(\tau_1, \tau_2) \times g_2(\tau_3, \tau_4)\{R_\xi(\tau_2-\tau_1) R_\xi(\tau_4-\tau_3)+R_\xi(\tau_2-\tau_3) R_\xi(\tau_1-\tau_4) +R_\xi(\tau_2-\tau_4) R_\xi(\tau_3-\tau_1)\} \tag{B−16}$$

$$E\{x^3\}=\int_{\tau_1} d\tau_1 \cdots\cdots \int_{\tau_4} d\tau_4 g_2(\tau_1, \tau_2) g_1(\tau_3) g_1(\tau_4)\{6R_\xi(\tau_3-\tau_1) R_\xi(\tau_4-\tau_2) +3R_\xi(\tau_2-\tau_1) R_\xi(\tau_4-\tau_3)\}+\{\int_{\tau_1} d\tau_1 \int_{\tau_2} d\tau_2 g_2(\tau_1, \tau_2) R_\xi(\tau_2-\tau_1)\}^3 +\int_{\tau_1} d\tau_1 \cdots\cdots \int_{\tau_6} d\tau_6 g_2(\tau_1, \tau_2) g_2(\tau_3, \tau_4) g_2(\tau_5, \tau_6)\{6R_\xi(\tau_2-\tau_1) R_\xi(\tau_6-\tau_3) \times R_\xi(\tau_5-\tau_4)+8R_\xi(\tau_3-\tau_1) R_\xi(\tau_5-\tau_2) R_\xi(\tau_6-\tau_4)\} \tag{B−17}$$

Accordingly we can obtain the statistical values of $x(t)$ as

$$\bar{x}=\int_\omega d\omega G_2(\omega, -\omega) S_\xi(\omega), \tag{B−18}$$

$$\sigma_x^2=\int_\omega d\omega |G_1(\omega)|^2 S_\xi(\omega)+2\int_{\omega_1} d\omega_1 \int_{\omega_2} d\omega_2 |G_2(\omega_1, \omega_2)|^2 \times S_\xi(\omega_1) S_\xi(\omega_2), \tag{B−19}$$

$$\mu\sigma_x^3=6\int_{\omega_1} d\omega_1 \int_{\omega_2} d\omega_2 G_1(-\omega_1) G_1(-\omega_2) G_2(\omega_1, \omega_2) S_\xi(\omega_1) S_\xi(\omega_2) +8\int_{\omega_1} d\omega_1 \int_{\omega_2} d\omega_2 \int_{\omega_3} d\omega_3 G_2(\omega_1, \omega_2) G_2^*(\omega_2, \omega_3) G_2(\omega_3, -\omega_1) \times S_\xi(\omega_1) S_\xi(\omega_2) S_\xi(\omega_3), \tag{B−20}$$

where $*$ denotes the complex conjugate.

Appendix C

Integral Evaluation to Instantaneous Probability Density Function of $x(t)$ as $x \to \infty$

From Eq. (2.4.2) the instantaneous probability density function of $x(t)$ may be rewritten as

$$p_x(x)=\lim_{R\to\infty} 1/2\pi \int_{-R}^{R} \exp(-i\theta x)\, \phi(\theta)\, d\theta. \tag{C−1}$$

First, we expand Eq. (C−1) to complex space. By regarding this integral as a complex integral and replacing $-i\theta$ by s we obtain the following form:

$$p_x(x)=\lim_{R\to\infty} 1/2\pi i \int_{-iR}^{iR} \prod_{j=1}^{\infty} (1+2\lambda_j s)^{-1/2} \exp\{sx+c_j^2 s^2/2(1+2\lambda_j s)\}\, ds. \tag{C−2}$$

Then without loss of generality it can be assumed that $\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$ and $0 > \lambda_{n+1} > \ldots > \lambda_m$.

For x $\to\infty$ only the positive eigenvalues are of interest. Thus in the case in which n is even we take the branch cuts from $-1/2\lambda_{2j-1}$ to $-1/2\lambda_{2j}$ ($j=1,\ldots,n/2$) along the real axis as shown in Fig. (C−1). If n is odd the branch-cut from $-1/2\lambda_n$ to $-\infty$ is added.

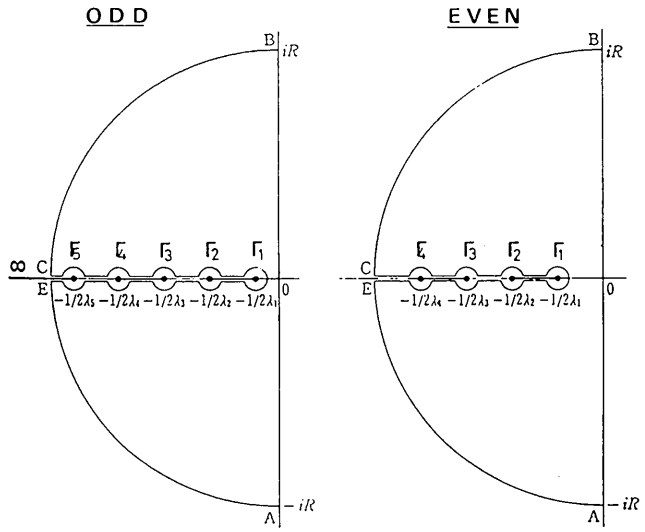

Fig. C-1 Contours of integration

Since $1+2\lambda_j s$ takes either positive value or negative value when s moves on the negative real axis, we define the brach of $1/\sqrt{1+2\lambda_j s}$ as

$$|1+2\lambda_j s|^{-1/2} \text{ for } 1+2\lambda_j s>0, \tag{C−3}$$

$$-i|1+2\lambda_j s|^{-1/2} \text{ for } 1+2\lambda_j s<0. \tag{C−4}$$

Then, the integrand of (C−1) is regular except for branch points $-1/2\lambda_j (j=1,\ldots,n)$.

Thus, by Cauchy's theorem the integral along ABC $\sum_j \Gamma_j$ E becomes zero. The integrals along each path are as follows:

1) Integral along BC

Setting $s=Re^{i\theta}(\pi/2\leqq\theta\leqq\pi)$ we have the following inequalities:

$$|\exp(c_j^2 s^2/2(1+2\lambda_j s))|\leqq \exp(c_j^2 R\cos\theta/4\lambda_j)\leqq 1,$$

$$|\exp(sx)| = \exp(xR\cos\theta) \leqq 1,$$

$$|1+2\lambda_j s|^{-1/2} \leqq |1-2\lambda_j R|^{-1/2}.$$

Thus the integral along BC becomes zero as $R \to \infty$.

(2) Integral along EA

It can be proved that the integral along EA also becomes zero as $R \to \infty$.

(3) Integral along the semi circle of Γ_j

Setting $s = \rho e^{i\theta} - 1/2\lambda_j (\pi \leqq \theta \leqq 0)$ we get the following inequalities:

$$|\exp(c_j^2 s^2/2(1+2\lambda_j s))| \leqq \exp\{c_j^2/2\lambda_j\rho(\rho^2\cos\theta - \rho/\lambda_j + \cos\theta/\lambda_j^2)\},$$

$$|\exp(sx)| \leqq \exp(\rho x\cos\theta)$$

$$|1+2\lambda_j s|^{-1/2} \leqq 1/\sqrt{2\lambda_j\rho}.$$

Thus, if ρ is taken such that it is equal to $c_j/(2\lambda_j\sqrt{2\lambda_j x})$, which is a stationary point, this integral becomes zero as $x \to \infty$.

Since the integrand of (C−1) is bounded at all points except for branch points, we obtain the following form as the evaluation of Eq. (C−1):

$$p_x(x) = 1/\pi \sum_j (-1)^{j-1} \int_{-1/2\lambda_{2j}}^{-1/2\lambda_{2j-1}} e^{sx} \prod_j |1+2\lambda_j s|^{-1/2} \times \exp\{c_j^2 s^2/2(1+2\lambda_j s)\}\, ds \qquad \textit{if } n \textit{ is even,} \quad (C-5)$$

$$p_x(x) = 1/\pi \sum_j (-1)^{j-1} \int_{-1/2\lambda_{2j}}^{-1/2\lambda_{2j-1}} e^{sx} \prod_j |1+2\lambda_j s|^{-1/2} \times \exp\{c_j^2 s^2/2(1+2\lambda_j s)\}\, ds + 1/\pi(-1)^n \int_{-\infty}^{-1/2\lambda_n} e^{sx} \times \prod_j |1+2\lambda_j s|^{-1/2} \exp\{c_j^2 s^2/2(1+2\lambda_j s)\}\, ds \qquad \textit{if } n \textit{ is odd} \quad (C-6)$$

If it is assumed that $\lambda_j/\lambda_1 (j = 2, \ldots n) < 1$, the main contribution to the integral (C−1) will be from the vicinity of the branch point λ_1.

Samely the integral as $x \to -\infty$ is also evaluated by taking the branch-cuts on positive real axis.

In the case of $n=2$ and $\lambda_1\lambda_2 < 0$ and $\lambda_j > c_j (j = 1, 2)$ the integral (C−1) is evaluated exactly as

$$p_x(x) = 1/2\pi \int_{-\infty}^{\infty} d\theta \exp(-ix\theta)/\sqrt{(1-2i\lambda_1\theta)(1-2i\lambda_2\theta)}$$
$$= 1/2\pi \sqrt{\lambda_1|\lambda_2|} \exp\{(\lambda_1 - |\lambda_2|)/4\lambda_1|\lambda_2|\} K_0\{|x|(\lambda_1 + |\lambda_2|)/4\lambda_1|\lambda_2|\}, \quad (C-7)$$

where K_0 is the modified Bessel function of the second kind.