Theoretical and Experimental Investigations of Interaction among Deep-Water Gravity Waves*

By

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summary

It is well known that most of the energy of sea waves, which causes a lot of damage to ships, off-shore structures and facilities on coasts, is supplied by winds blowing over the ocean.
However, if a wind strong enough to generate gravity waves stops, the gravity waves, far from dying out rapidly, will continue to run straight on until they fetch up against something. Once waves have escaped from the wind that made them, they can run for days with very little loss of energy. Therefore, they travel long distance without the influence of winds. Moreover, these wave elements change their properties owing to the mutual interaction during this stage. Accordingly, to understand the nature of sea waves, besides studying the mechanism of wind-wave interaction, it is also imperative to clarify the characteristics of propagation of an individual wave train. In this paper, we deal with the nonlinear dynamics of the deep-water gravity waves and apply it to the experiment to interpret the results concerning the mutual interaction among waves.

The contents of each Chapter are as follows. In Chapter 1, we review the basic theory of water waves and formulate the problems from

* Received on May 8, 1989
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the point of view of a singular perturbation method. In the following two Chapters, experimental and numerical studies concerning the particular condition of the resonant wave interactions are described. In Chapter 2, long term evolution of tertiary resonant waves are detected experimentally and the direction of propagation of the resonant wave is also obtained for the first time by aid of the cross-spectral analysis. The purpose of the observations is twofold: to examine quantitatively the evolution of the amplitude modulation and to test the validity of weakly non-linear wave theory (Zakharov equation) for the asymptotic behavior of resonant waves by comparing the predicted and the observed properties of the waves.

In Chapter 3, the Zakharov's integro-differential equation is solved numerically and it is shown that the experimental data agree with the solutions in the case of comparatively small wave steepness. Calculations are also performed to determine the dependence of the maximum amplitude of the resonant wave upon the amplitude of primary waves. In Chapter 4, comparisons of the experimental results with theories are made both for classical and that by Zakharov. It is concluded that the former is insufficient to explain quantitatively the long term evolution of the tertiary resonant wave and that the latter model of non-linear water waves is applicable for describing the propagation of sea waves because of fairly good agreement of the theory with data.

Several theoretical remarks, including the analytical investigation into a particular solution of the discretized Zakharov equation, are offered in Appendices.

CONTENTS

CHAPTER 1 NON-LINEAR DYNAMICS OF WATER WAVES

1. 1 Foreword

It is well known that the work of Stokes titled "On the theory of oscillatory waves" in 1847 is substantially the first study of the non-linear property of water waves. In this pioneering paper, he gave a stationary solution of a train of deep-water gravity waves by aid of the power series expansion with respect to wave steepness. Many important properties of non-linear waves, such as the dispersion relation dependent on amplitude, the existence of highest limit of wave and the drift motion of particles in a wave were shown in his work. Besides the above mentioned theory, the Trochoidal wave, an exact particular solution of water wave, found by Gerstner(1809), had been applied in the field of naval architecture for a long time. These basic solutions are the most important ones in the non-linear water wave theory.

On the other hand, the researches concerning the description of ocean waves have been developed in a somewhat different manner. In this field, the subject is divided into two main parts. One is to investigate the mechanisms of wave generation by wind. The other is to describe the actual configuration of ocean surface properly.

In this paper, we deal mainly with the latter problem. The study of the scientific description of sea waves was started at the beginning of 1950s with the work of Pierson(1952) who introduced the concepts of stochastic processes and of spectrum to oceanography. His investigation for wave forecasting has been developed considerably by aid of electronic computers. However, from the theoretical point of view, there is enough ground for controversy in his method. Pierson, Neumann & James (1955) assumed that the fluctuation of the ocean surface is composed of many infinitesimal wave trains which travel independently to each other in their own directions. According to this assumption, the spectrum of sea wave is recognized as a distribution function of the energy of component waves. On the contrary, the stochastic variation of surface displacement, its velocity or acceleration satisfies the Gaussian distribution and the moments can be determined by the spectrum. So far as we admit the linear wave theory, there would be no problem conceptually.

Once we draw attention to the non-linear properties of water waves and consider them in the framework of the PNJ method, most of the concepts would become ambiguous. However, no one could have extended the theory to contain the non-linear characteristics of waves in the ages of

1950s, because the theory of non-linear waves had been no more improved than that established in 19th century. To overcome this difficulty, there appeared many papers concerning the non-linear theory with regard to multiple component wave system since 1960s. We mention some of them, in relation to this paper such as Tick(1959) and Hamada(1966), which are the second order theory for random wave field. Huang & Tung(1976), Weber & Barrick(1977), Barrick & Weber(1977), Masuda, Mitsuyasu & Kuo(1979) and Mitsuyasu, Kuo & Masuda(1979) dealt with the third order random wave field although they did not take the energy transfer among component waves into account except for the change of wave velocity. The last one involves the experimental verification in a wind-wave flume. In earlier, Phillips(1960) proposed the theory for accounting the energy transfer between wave components however his mathematical formulation contained a singular property and did not offer the solution describing the long-time evolution of resonant waves. Benney(1962) gave the equations which describe the long-time behavior of four waves for the first time.

Zakharov(1968) derived the equation governing the mutual interaction among deep-water gravity waves of arbitrary number of components in the most purely theoretical point of view. Stiassnie & Shemer(1984) rederived it by somewhat elementary method with using Fourier transform technique. They are most closely related ones to the present paper. In this Chapter, we reexamine those works and discuss the non-linear dynamics of water waves in the unified point of view. Some precise study concerning the characteristics of Zakharov equation containing the numerical and analytical solution will be discussed in Chapter 3 and in Appendices.

In addition, we also mention the book "The Dynamics of the Upper Ocean" written by Phillips(1977) as the most excellent description and the basic results of sea waves. The simple and fine explanations are referred in the articles written by Nagata(1970) and Taira(1975).

1. 2 Basic Equations

In this Chapter, we assume in regard to hydrodynamic natures of water waves that the viscosity is neglected (perfect fluid), and that the motion is irrotational and the compressibility of the fluid is neglected. Capillarity and air motion above the surface of fluid are not taken into account. The density of water is assumed not to change temporally and spatially.

We deal the present problem as, in three dimensional space, that is, two dimensional sufficiently large horizontal surface which is uniform and isotropic. The depth of sea is infinite. We also assume that the amplitude of the wave is small but finite.

From the assumption of irrotational motion, there exists the velocity potential ϕ in the fluid. By the assumption of incompressibility, the equation of continuity is satisfied

$$\nabla^2 \phi = 0 \qquad (1-1)$$

in the interior of the fluid. Here, we take the coordinate system as x- and y-axes in horizontal and z-axis in the vertical upwards direction respectively. At the fluid surface ($z = \eta$) the kinematic boundary condition

$$\frac{\partial \eta}{\partial t} + \nabla_h \phi \nabla_h \eta = \frac{\partial \phi}{\partial z} \qquad (1-2)$$

and the dynamic boundary condition

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \nabla \phi \nabla \phi = -g \eta \qquad (1-3)$$

are satisfied. Where, η denotes the displacement of the surface and g represents the acceleration due to the gravity. The operator ∇_h means the horizontal components of gradient operator ∇. From the assumption, the density of water is constant so that it does not appear in these equations. The difficulty of the problems on water waves lies on the fact that the above equations (1 − 2) and (1 − 3) are both non-linear and the form of the boundary η is not determined ab initio but is an unknown variable. Finally, from the assumption in the limit $z \to -\infty$,

$$\nabla \phi \to 0 \qquad (1-4)$$

is required.

1. 3 Some Aspects on Classical Theory

On the basis of the general theory in hydrodynamics, we restrict ourselves to the problem of non-linear resonant wave interaction. Phillips(1960) discovered that in the third approximation, it is possible for a transfer of energy to take place from three primary waves (of wave-numbers $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_4$) to a fourth wave (of wave-number $\mathbf{k}_3$) in such a way that the amplitude of the fourth wave increases linearly with time. Thus, although the fourth wave amplitude at first is very small(being of the third order) it may grow in time so as to be comparable with the three primary waves. The condition for this is that the wave-numbers $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, $\mathbf{k}_4$ and frequencies ω_1, ω_2, ω_3, ω_4 each satisfy the dispersion relation:

$$\omega_i^2 = g\,|\,\mathbf{k}_i\,| \qquad (i=1,2,3,4)\ , \qquad (1-5)$$

and that

$$\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 \pm \mathbf{k}_4 = 0\ , \qquad \omega_1 \pm \omega_2 \pm \omega_3 \pm \omega_4 = 0\ , \qquad (1-6)$$

with the same combination of signs in each case.

At first, we explain briefly the theoretical results obtained by the direct use of a perturbation technique (REGULAR PERTURBATION) to the basic equations. Longuet-Higgins(1962) has analysed this problem in the case that $\mathbf{k}_1 = \mathbf{k}_4$, $\omega_1 = \omega_4$, the condition (1 − 6) turns out to be

$$2\,\mathbf{k}_1 - \mathbf{k}_2 = \mathbf{k}_3, \qquad 2\,\omega_1 - \omega_2 = \omega_3\ . \qquad (1-7)$$

Phillips(1960) showed that in the case that resonance condition (1 − 7) is satisfied, wave-number $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ should be correlated each other as shown in Fig− 1 − 1 . In the special condition that $\mathbf{k}_1 \perp \mathbf{k}_2$, $\gamma = \omega_1 / \omega_2$ would be $\gamma = \gamma_0 = 1.736\cdots$.

The velocity potential ϕ and surface displacement η are assumed to be expressed in expanded series such that

$$\phi = (\alpha\,\phi_{10} + \beta\,\phi_{01}) + (\alpha^2\phi_{20} + \alpha\,\beta\,\phi_{11} + \beta^2\phi_{02}) +$$
$$+ (\alpha^3\phi_{30} + \alpha^2\beta\,\phi_{21} + \alpha\,\beta^2\phi_{12} + \beta^3\phi_{03}) + \cdots\cdots \qquad (1-8-1)$$

and

$$\eta = (\alpha\,\eta_{10} + \beta\,\eta_{01}) + (\alpha^2\eta_{20} + \alpha\,\beta\,\eta_{11} + \beta^2\eta_{02}) + \cdots\cdots$$
$$+ (\alpha^3\eta_{30} + \alpha^2\beta\,\eta_{21} + \alpha\,\beta^2\eta_{12} + \beta^3\eta_{03}) + \cdots\cdots \qquad (1-8-2)$$

with α and β being independent small quantities representing the wave steepness of each wave. Substituting (1 − 8 − 1) and (1 − 8 − 2) to the basic equations (1 − 2) and (1 − 3) , the calculations were carried out up to the third order terms. We only pay attention to the term ϕ_{21}, because it represents the tertiary resonant wave to be considered here. Longuet-Higgins & Smith(1966) gave solution ϕ_{21} at $z=0$ as

$$\phi_{21} = -\frac{K}{g\,\delta k}\sin(\delta k x)\sin\{(k_0+\delta k)x-\omega_3 t\},$$
(1 − 9)

under a slightly extended conditions that

$$2k_1 - k_2 = k_3, \quad 2\omega_1 - \omega_2 \sim \omega_3 . \qquad (1-10)$$

In (1 − 10), equality might not be satisfied strictly for frequencies. In the equation (1 − 9), K is the growth rate and expressed as

$$K = (a_1 k_1)^2 a_2 k_2 g^2 \omega_3^{-1} G,$$

with non-dimensional coefficient G. k_0 equals to ω_0^2/g where ω_0 is defined as $\omega_0 = 2\omega_1 - \omega_2$ and $2\delta k = k_3 - k_0$. δk and $\delta\gamma = \gamma - \gamma_0$ are correlated as

$$\frac{2\delta k}{k_3} = -\left(\frac{4}{2\gamma_0 - 1} - \frac{8\gamma_0^3}{4\gamma_0^4 + 1}\right)\delta\gamma . \qquad (1-11)$$

From the form of (1 − 9), we can recognize that amplitude of tertiary wave varies slowly with x when $\delta\gamma \neq 0$. If $\delta\gamma = 0$, the solution ϕ_{21} in (1 − 9) appears to be infinite, however in such a limiting case, it reduces to

$$\phi_{21} \rightarrow -\frac{Kx}{g}\frac{\sin\delta k x}{\delta k x} \sim \frac{Kx}{g} . \qquad (1-12)$$

Thus, tertiary wave grows linearly with x. Transforming it to the wave amplitude a, the maximum amplitude a_{3M} to be realized by tertiary wave is obtained as

$$a_{3M} = \frac{(a_1 k_1)^2 a_2 k_2}{|\delta k|} G . \qquad (1-13)$$

The constant G is given by Longuet-Higgins(1962) as 0.442. In order to express a_{3M} by the explicit function of $\delta\gamma$, we eliminate δk in (1 − 9) by using (1 − 11), we have a classical approximation

$$\frac{a_{3M}}{a_2} \simeq 0.491 \frac{(a_1 k_1)^2}{|\delta\gamma|} . \qquad (1-14)$$

1.4 Expansion Procedure of the Solution

In this section, we derive the equation which governs the interaction among components of gravity wave system. The method of derivation is essentially different from the classical one as explained in §1.3 and is applicable to developing stage of non-linear interactions. In order to consider generally the two-dimensional multiple component wave system, the velocity potential ϕ and sea surface displacement η in the basic equations (1−1) ~ (1−4) are expressed as spatial Fourier serieses of the forms,

$$\phi(\mathbf{r}, z, t) = \Sigma_k A(\mathbf{k}, z, t) \exp(i\mathbf{k}\mathbf{r}) \qquad (1-15)$$

and

$$\eta(\mathbf{r}, t) = \Sigma_k B(\mathbf{k}, t) \exp(i\mathbf{k}\mathbf{r}) . \qquad (1-16)$$

From the pure mathematical point of view, Fourier integral or Fourier-Stieltjes integral representation must be used, but according to Weber & Barrick(1977), in the case of the assumption that the horizontal area considered here is finite though sufficiently larger than typical wavelength, equations (1−15), (1−16) are hold good. As ϕ satisfies the conditions (1−1) and (1−4), velocity potential ϕ has the form

$$\phi(\mathbf{r}, z, t) = \Sigma_k A(\mathbf{k}, t) \exp(kz + i\mathbf{k}\mathbf{r}) . \qquad (1-17)$$

In the process from now on, the several points explained in the following subsections should be considered carefully;

1.4.1 Treatment of the boundary conditions on unfixed surface $z=\eta$

When we treat the basic equations (1−2) and (1−3) on the surface $z=\eta$, all the terms containing derivatives of ϕ are proportional to $\exp(k\eta)$. We assume the wave steepness $k\eta \ll 1$ and use the Taylor expansion

$$\exp(k\eta)=1+k\eta+(1/2)k^2\eta^2+(1/6)k^3\eta^3+\cdots\cdots.$$

For example, ϕ_t is calculated in the following way.

First, we differentiate (1 − 1 7) with respect to t and insert η in place of z. Next, we use the Taylor expansion of the exponential function above and substitute the expression (1 − 1 6) into the powers of η. We finally obtain in the form of spatial Fourier series as

$$\frac{\partial\phi}{\partial t}=F_1+F_2+F_3+\cdots\cdot\ .$$

Where, F_n $(n=1,2,3,\cdots\cdot)$ represents the n-th order quantities as

$$F_1=\Sigma_k A_t(\mathbf{k})\exp(i\mathbf{k}\mathbf{r}),$$

$$F_2=\Sigma_k \exp(i\mathbf{k}\mathbf{r})[\Sigma_{k1}k_1A_t(\mathbf{k}_1)B(\mathbf{k}-\mathbf{k}_1)]$$

$$F_3=\tfrac{1}{2}\Sigma_k \exp(i\mathbf{k}\mathbf{r})[\Sigma_{k1}B(\mathbf{k}-\mathbf{k}_1)\{\Sigma_{k2}k_2{}^2A_t(\mathbf{k}_2)$$

$$B(\mathbf{k}_1-\mathbf{k}_2)\}],$$

and

$$\cdots\cdots\cdots.$$

Calculating $\nabla\phi$ in the similar manner, the results are substituted into (1 − 2) and (1 − 3). Utilizing the orthogonality property of Fourier series, we can transform the basic equations to the simultaneous differential equation with respect to A and B. We can finally obtain the results up to the third order of A and B as follows

$$B_t(\mathbf{k})-kA(\mathbf{k})=\Sigma_{k1}\{\mathbf{k}_1\cdot(\mathbf{k}-\mathbf{k}_1)+k_1{}^2\}A(\mathbf{k}_1)$$

$$B(\mathbf{k}-\mathbf{k}_1)+\Sigma_{k1}B(\mathbf{k}-\mathbf{k}_1)\Sigma_{k2}\{k_2\mathbf{k}_2\cdot(\mathbf{k}_1-\mathbf{k}_2)+$$

$$\tfrac{1}{2}k_2{}^3\}A(\mathbf{k}_2)B(\mathbf{k}_1-\mathbf{k}_2) \qquad (1-18)$$

and

$$A_t(\mathbf{k})+gB(\mathbf{k})=\frac{1}{2}\Sigma_{k1}\{\mathbf{k}_1\cdot(\mathbf{k}-\mathbf{k}_1)-k_1|\mathbf{k}-\mathbf{k}_1|\}$$

$$A(\mathbf{k}_1)A(\mathbf{k}-\mathbf{k}_1)-\Sigma_{k1}k_1A_t(\mathbf{k}_1)B(\mathbf{k}-\mathbf{k}_1)-$$

$$\frac{1}{2}\Sigma_{k1}B(\mathbf{k}-\mathbf{k}_1)\Sigma_{k2}k_2{}^2A_t(\mathbf{k}_2)B(\mathbf{k}_1-\mathbf{k}_2)+$$

$$\Sigma_{k1}B(\mathbf{k}-\mathbf{k}_1)\Sigma_{k2}\{k_2\mathbf{k}_2\cdot(\mathbf{k}_1-\mathbf{k}_2)-k_2{}^2|\mathbf{k}_1-\mathbf{k}_2|\}$$

$$A(\mathbf{k}_2)A(\mathbf{k}_1-\mathbf{k}_2). \qquad (1-19-1)$$

Here, suffix t means the time derivatives and from now on, we use the expression $A(\mathbf{k})$ instead of $A(\mathbf{k},t)$ omitting independent variable t. Except for Phillips(1960), Zakharov(1968) and Stiassnie & Shemer(1984), theories by the other authors were restricted that A, B are periodic functions so that the equations were reduced merely to algebraic relations (in fact, setting A, $B\propto\exp(-i\omega t)$, we could show that equations (1－18) and (1－19－1) reduce to those in Weber & Barrick(1977) after some simple algebraic manipulation).

1．4．2 Transforming the equations to apply the singular perturbation method

In order to arrange the equations to apply the SINGULAR PERTURBATION METHOD, time derivative A_t in the right-hand side of (1－19－1) has to be eliminated. At first, we neglect terms higher than second order and we have

$$A_t(\mathbf{k})\fallingdotseq -gB(\mathbf{k})$$

in the first order. Substituting this into (1－19) iteratively, we obtain the second order approximation as,

$$A_t(\mathbf{k})\fallingdotseq -gB(\mathbf{k})+\frac{1}{2}\Sigma_{k1}\mathbf{k}_1\cdot(\mathbf{k}-\mathbf{k}_1)A(\mathbf{k}_1)$$

$$A(\mathbf{k}-\mathbf{k}_1)+\Sigma_{k1}g\,k_1B(\mathbf{k}_1)B(\mathbf{k}-\mathbf{k}_1).$$

Using them in (1－19) again, up to the third order it is transformed into

$$A_t(\mathbf{k}) + gB(\mathbf{k}) = \tfrac{1}{2}\Sigma_{k1}\{\mathbf{k}_1\cdot(\mathbf{k}-\mathbf{k}_1) - k_1|\mathbf{k}-\mathbf{k}_1|\}$$

$$A(\mathbf{k}_1)A(\mathbf{k}-\mathbf{k}_1) + \Sigma_{k1} g k_1 B(\mathbf{k}_1)B(\mathbf{k}-\mathbf{k}_1) -$$

$$\Sigma_{k1} B(\mathbf{k}-\mathbf{k}_1)\Sigma_{k2} g k_2(k_1 - \tfrac{1}{2}k_2)B(\mathbf{k}_2)B(\mathbf{k}_1-\mathbf{k}_2)$$

$$-\Sigma_{k1} B(\mathbf{k}-\mathbf{k}_1)\Sigma_{k2}(\tfrac{1}{2}k_1 - k_2)\{\mathbf{k}_2\cdot(\mathbf{k}_1-\mathbf{k}_2) -$$

$$k_2|\mathbf{k}_1-\mathbf{k}_2|\}A(\mathbf{k}_2)A(\mathbf{k}_1-\mathbf{k}_2). \qquad (1-19-2)$$

The combination (1 − 1 8) and (1 − 1 9 − 2) reduces to the equations of harmonic oscillation in the limit A, B→0.

1.4.3 Technique for eliminating the variable A or B with the consideration that ϕ and η are real quantities

A and B are the Fourier coefficients of the velocity potential ϕ and the surface displacement η. As ϕ and η are real numbers, A and B must be complex numbers whose dependence on $\mathbf{k}$ have the antisymmetric nature

$$A(\mathbf{k}) = A^*(-\mathbf{k}) \qquad (1-20-1)$$

and

$$B(\mathbf{k}) = B^*(-\mathbf{k}). \qquad (1-20-2)$$

Where, A^* is a complex conjugate of A. Thus, we can introduce such a complex variable Z that

$$i\alpha_k A(\mathbf{k}) = Z(\mathbf{k}) - Z^*(-\mathbf{k}) \qquad (1-21-1)$$

$$\beta_k B(\mathbf{k}) = Z(\mathbf{k}) + Z^*(-\mathbf{k}) \qquad (1-21-2)$$

for the reason that the relations (1 − 2 0 − 1) and (1 − 2 0 − 2) are satisfied automatically. In these relations, α_k and β_k are the real constants dependent only upon the magnitude of the wave-number k.

If we execute the transformation (1 − 2 1), we can deal two unknowns A and B as in one unknown Z formally. The resultant equation of Z again reduces to that of harmonic oscillation only if the constants α_k and β_k satisfy the relation

$$k\alpha_k^{-2} = g\beta_k^{-2} \quad . \qquad (1-22)$$

In this paper, according to Stiassnie & Shemer(1984),

$$\alpha_k^2 = 2\,(k/g)^{1/2},\quad \beta_k^2 = 2\,(g/k)^{1/2} \qquad (1-23)$$

is adopted.

The equations (1 − 18) and (1 − 19 − 2) are transformed by means of (1 − 21). If Z^*_t is eliminated in these equations, the linear part of Z^* also vanishes owing to (1 − 22). Thus, the equation with respect to Z is obtained in the following such that

$$iZ_t - (gk)^{1/2} Z = J(\mathbf{k},\ Z)$$

where, $J(\mathbf{k},\ Z)$ is yielded by $i\alpha^{-1}$ times of the right-hand side of (1 − 18) minus β^{-1} times of the right-hand side of (1 − 19 − 2).

Explicit form of the equation is

$$\begin{aligned} iZ_{kt} - \omega_k Z_k = {} & \sum_{k1,k2} H^{(1)}(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2)\, Z_{k1} Z_{k2} + \sum_{k1,k2} H^{(2)}(\mathbf{k}, \\ & \mathbf{k}_1,\ \mathbf{k}_2)\, Z_{k1} Z_{k2}^* + \sum_{k1,k2} H^{(3)}(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2) \\ & Z_{k1}^* Z_{k2}^* + \sum_{k1,k2,k3} F^{(1)}(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3)\, Z_{k1} Z_{k2} \\ & Z_{k3} + \sum_{k1,k2,k3} F^{(2)}(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3)\, Z_{k1}^* Z_{k2} Z_{k3} + \\ & \sum_{k1,k2,k3} F^{(3)}(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3)\, Z_{k1}^* Z_{k2}^* Z_{k3} + \\ & \sum_{k1,k2,k3} F^{(4)}(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3)\, Z_{k1}^* Z_{k2}^* Z_{k3}^* \quad . \end{aligned}$$

$$(1-24)$$

Where, ω_k is angular frequency given by $\omega_k = (gk)^{1/2}$ which is the dispersion relation of deep-water gravity waves. Equation (1 − 24) is the MODE COUPLING EQUATION to describe the propagation of finite amplitude water waves discussed in this paper. The concrete expression of the coefficients $H^{(n)}(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2)$ and $F^{(n)}(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3)$ are presented in Appendix I. By use of the complex amplitude Z, surface elevation η is represented as

$$\eta(\mathbf{r}, t) = \tfrac{1}{2}\Sigma_k \beta_k^{-1}[Z(\mathbf{k}, t) + Z^*(-\mathbf{k}, t)] \exp(i\mathbf{k}\mathbf{r}). \tag{1-25}$$

1. 5 Perturbation Method and Zakharov Theory

In this section we apply the singular perturbation method to analyse non-linear equation like (1 - 2 4) in contrast to the regular perturbation method used in § 1. 3. As discussed briefly in § 1. 3, the application of the regular perturbation method to non-linear equation results in the solution infinitely increasing with time t. This fact means that the method is not suitable to express the long-time variation of the solutions. Therefore, to avoid such a difficulty and to obtain the long-time evolution of solution, we adopt here the MULTIPLE SCALE METHOD, a sort of the singular perturbation method. The essence of the method lies on the technique introducing the slowly varying independent variables. We execute this procedure somewhat more systematically than Zakharov(1968)or Stiassnie & Shemer(1984). This method is applicable only to the non-linear equations of the form discussed in § 1. 4. 2 of the preceding section (Bogoliubov & Mitropolskii(1965) called them quasi-linear equation) .

Now, we introduce a small parameter ε and expand Z as

$$Z = \varepsilon Z^{(1)} + \varepsilon^2 Z^{(2)} + \varepsilon^3 Z^{(3)} + \cdots\cdots. \tag{1-26}$$

Furthermore, we introduce a group of independent variables $T_n = \varepsilon^n t$ instead of t. Then, Z is regarded as the function not only of t but also of T_n (n=1, 2, 3, $\cdots$, $T_0 = t$) . So, the equation (1 - 2 4) becomes a partial differential equation. Differential operator is also expanded as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2} + \varepsilon^3\frac{\partial}{\partial T_3} + \cdots\cdots. \tag{1-27}$$

We substitute (1 - 2 6) and (1 - 2 7) into (1 - 2 4) and rearrange it with respect to the power series of ε. Then, for the first order,

$$i Z_k{}^{(1)}{}_{T0} - \omega_k Z_k{}^{(1)} = 0 \tag{1-28}$$

is obtained. If we take up to the second order of ε, we have

$$i Z_k{}^{(2)}{}_{T0} - \omega_k Z_k{}^{(2)} = - i Z_k{}^{(1)}{}_{T1} +$$

$$\sum_{k1, k2} H^{(1)}(k, k_1, k_2) Z_{k1}{}^{(1)} Z_{k2}{}^{(1)} +$$

$$\sum_{k1, k2} H^{(2)}(k, k_1, k_2) Z_{k1}{}^{(1)} Z_{k2}{}^{(1)*} +$$

$$\sum_{k1, k2} H^{(3)}(k, k_1, k_2) Z_{k1}{}^{(1)*} Z_{k2}{}^{(1)*}$$

(1－29)

Assuming the periodic solution of T_0, the first order equation is immediately solved as

$$Z_k{}^{(1)} = X_k{}^{(1)} \exp(- i \omega_k T_0). \qquad (1-30)$$

$X_k{}^{(1)}$ is an arbitrary function which is independent of T_0. Substituting (1－30) to the second order of (1－29), we have

$$i Z_k{}^{(2)}{}_{T0} - \omega_k Z_k{}^{(2)} = - i X_k{}^{(1)}{}_{T1} \exp(- i \omega_k T_0) +$$

$$\sum_{k1, k2} H^{(1)}(k, k_1, k_2) X_{k1}{}^{(1)} X_{k2}{}^{(1)} \exp\{- i (\omega_{k1} + \omega_{k2}) T_0\} +$$

$$\sum_{k1, k2} H^{(2)}(k, k_1, k_2) X_{k1}{}^{(1)} X_{k2}{}^{(1)*} \exp\{- i (\omega_{k1} - \omega_{k2}) T_0\} +$$

$$\sum_{k1, k2} H^{(3)}(k, k_1, k_2) X_{k1}{}^{(1)*} X_{k2}{}^{(1)*} \exp\{ i (\omega_{k1} + \omega_{k2}) T_0\}.$$

(1－31)

In this equation, we should notice to combinations for the first term of the right-hand side with another term, say the second term, in the right-hand side. These terms are summed up to the following way as

$$- [i X_k{}^{(1)}{}_{T1} - \sum_{k1, k2} H^{(1)}(k, k_1, k_2) X_{k1}{}^{(1)} X_{k2}{}^{(1)}$$

$$\exp\{- i (\omega_{k1} + \omega_{k2} - \omega_k) T_0\}] \exp(- i \omega_k T_0).$$

(1－32)

If under the summation Σ_k, two conditions

$$\mathbf{k}_1+\mathbf{k}_2=\mathbf{k} \quad \text{and} \quad \omega_{k1}+\omega_{k2}\sim\omega_k \tag{1-33}$$

are simultaneously satisfied, then the time dependence of (1 − 3 2) are proportional to exp $(-i\,\omega_k T_0)$. If there exists such a term in the equation, the soultion $Z_k^{(2)}$ of (1 − 3 1) is known to diverge with respect to time T_0. To avoid the divergence of the solution, we should recognize the whole sum of the terms in [] of (1 − 3 2) to be zero. In other words, under the condition (1 − 3 3),

$$i\,X_k{}^{(1)}{}_{T1}-\sum_{k1,\,k2}H^{(1)}\,(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2)\ X_{k1}{}^{(1)}X_{k2}{}^{(1)}=0 \tag{1-34}$$

should be satisfied. By virtue of (1 − 3 4), $X_k{}^{(1)}$ is determined with respect to T_1. The case of another combination is discussed in a similar manner. If the conditions (1 − 3 3) are not satisfied simultaneously at all, only the equation

$$i\,X_k{}^{(1)}{}_{T1}=0$$

is required. It means that $X_k{}^{(1)}$ is independent of T_1. In reality, as for the deep-water gravity waves, the relations (1 − 3 3) are not satisfied (see, for example Kinsman(1965)) so that $X_k{}^{(1)}$ is constant up to this order. By use of this result, the equation (1 − 3 1) is easily solved for $Z_k{}^{(2)}$.

As the next step, the solution $Z_k{}^{(2)}$ is substituted in the third order equation and the caluculation is executed in the similar manner, then the conditions corresponding to (1 − 3 3) are described as

$$\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}+\mathbf{k}_3 \quad \text{and} \quad \omega_{k1}+\omega_{k2}\sim\omega_k+\omega_{k3}\,. \tag{1-35}$$

These are called the RESONANCE CONDITION of deep-water gravity waves. The condition that the solution is valid for the long time is determined by a similar equation to (1 − 3 4) and it represents the T_2 dependence of the first order solution $X_k{}^{(1)}$. This is known as ZAKHAROV TYPE EQUATION and is discussed in Chapter 3 of this paper. The properties of the equation are precisely interpreted in AppendixⅢ～Ⅸ.

There could be many other derivations to obtain the mutual interaction equation for water waves. The most formal treatment of the theory by use of CANONICAL THEORY is briefly interpreted in Appendix II. These treatment was applied to the stochastic problems in wind wave field by West(1981) slightly different manner from that discussed in this paper.

CHAPTER 2 EXPERIMENT IN A WAVE BASIN

2. 1 Foreword

In this Chapter, the experiment of non-linear resonant wave interactions performed in the SHIP EXPERIMENT BASIN of the Ship Research Institute (see, Tomita & Sawada(1987)) is described.

Long-time evolution of tertiary resonant waves has not yet been observed in a wave flume. Hence, the experiment is carried out to detect the evolution at the locations spreading widely in a flume. The investigation is performed to find what amount of interaction occurs under several conditions being prescribed.

In this experiment, we choose the simplest feature for examining the resonant interaction phenomena of growing up of the tertiary wave by the perpendicularly intersecting two trains of waves generated with the wave-makers. According to the theory of resonant wave interaction discussed in Chapter 1, the resonance takes place under the condition

$$\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 + \mathbf{k}_4 = 0, \quad \omega_1 - \omega_2 - \omega_3 + \omega_4 \sim 0 \quad . \qquad (2-1)$$

In particular, in this experiment, $\mathbf{k}_1 = \mathbf{k}_4$, $\omega_1 = \omega_4$ and $\mathbf{k}_1$ is orthogonal to $\mathbf{k}_2$. In this case, (2 − 1) are solved with respect to $\gamma = \omega_1 / \omega_2$ so that the exact resonance condition is given by $\gamma = 1.736\cdots$. Under this condition, the short time behavior of tertiary wave was discussed by Phillips(1960) and Longuet-Higgins(1962) theoretically, to which we refered briefly in Chapter 1. The experimental studies were also made by Longuet-Higgins & Smith(1966) and McGoldrick, Phillips, Huang & Hodgson(1966) in the smaller wave tanks with the sizes of not exceeding 3 meters square. All these investigations mentioned above were confined to discuss the initial growth of tertiary wave and to verify its growth rate. On the contrary, in our experiment, the observations of long term development of tertiary waves are carried out by use of a comparatively large basin.

Several remarkable results are obtained in this experiment. Above all, it is confirmed that the large amplitude resonant waves which are comparable to that of primary waves appear at the longer fetches than those in previous experiments. These resonant waves travel in the direction which the theory predicts. Moreover, resonant waves are directly observed by photo as an evidence of their existings, for the first time in the field of pure gravity-waves. We examine in the next place the short fetch behavior of resonant wave growth to compare it with those of the papers above. Finally, we advance further to the long fetch behavior of resonant waves and find the recurrence properties (see for example Waters & Ford(1966)) of interaction among gravity waves. The results are compared with the theory given by Zakharov(1968) which could be applied to the case of this experiment.

In addition to these studies, the observation of the resonant interacting wave system by photographic technique was recently carried out by Strizhkin & Ralentnev(1986)in real open ocean.

2. 2 Description of the Apparatus

As is seen in Fig－2－1, the basin has the size of 80 m in length, 80 m in width and 4.5m in depth. Two wave-makers are installed in the adjacent sides of the basin. The first one is plunger type of 54 m in width drived with 24 sets of 6kw minertia motor, the second one is flap type of 80 m in width drived with two sets of 90kw DC motor. Many trains of waves advancing in different directions can be generated with them. There exists absorbing artificial beaches at the opposite side of each wave-maker. The precise specification of the facility is explained in Shiba(1961) and Takaishi et.al.(1973a, b).

All wave gauges are capacitance type with nominal precision of ±1% and are arranged on the wire rope suspended above the surface of the basin. Each probe is fixed vertically by anchor settled on the bottom. Three examples of the measurement are shown in Fig－2－2. In this figure, the cases that (a) first wave only, (b) second wave only and (c) both waves are simultaneously generated are shown. Upper six rows represent the water surface variations detected by each probe, lower two rows are for the strokes of both wave-makers.

Data collection system is schematically drawn in Fig－2－3. Output signals are sent into the recorder and they are also transferred into disquet of a desktop computer through AD converter. Length of each run is limited to 200 seconds for suppressing the effect of wave reflection. Data are digitized every 0.1 second so that we keep the Nyquist frequency as 5Hz. This is sufficiently large value for the present problem.

The effect that the strokes of the wave-makers are finite is considered to be negligible in this experiment. We set the strokes as small as possible to avoid the unfavourable effects of wave breaking, second order wave generation and/or third order wave instability. Nevertheless, diffraction is not completely neglected because the total widths of the partitions are not infinite (diffraction effect was examined by Ishida et.al.(1980) for this basin applying the wave making theory). In order to avoid the ambiguity that the height of mechanically generated waves is not constant along its crest, average values for the waves are used.

2.3 Method of Experiment

Having described the experimental apparatus, let us now turn to the method of measurement. The measurements are executed on two sorts of arrangement of wave gauges shown in Fig-2-4 (Case I) and Fig-2-5 (Case II). The former is used to reexamine the short term behavior of tertiary wave which was carried out by McGoldrick's experiment and for the first time to detect the direction of propagation of tertiary resonant wave. The latter is used for the measurement of long term developement of tertiary resonant waves. At each measurement, the amplitudes of three component waves which would simultaneously exist in the basin are estimated by the power spectral analysis by means of FFT as follows

$$A_k^2/2 = (P_{k-1} + P_k + P_{k+1})\Delta f \quad . \qquad (2-2)$$

In this equation, A_k and P_k denote the amplitude and component energy density corresponding to the frequency $f = k\Delta f$ ($\Delta f = 0.0098\ sec^{-1}$). As is well known in spectral analysis, the energy at single frequency is apt to disperse to its neighbourhoods caused by that the length of data is finite. The precision of this method is tested by aid of dummy data made with electric oscillator. By this test a single component of energy is apparntly broadened in width of $\pm 10\Delta f$ band at the attenuation of -30dB. Considering the noise property of real data, the band width of $3\Delta f = 0.029\ sec^{-1}$ is adopted as shown in (2-2). By using (2-2), restoration ratio of the test data is about 97%.

Our experimental situation and the size of facility lie between most of smaller-scale indoor laboratories and large natural sea field. So the unfavourable affections caused by viscosity and capillarity of water are negligible. All the works are conducted during calm weather, because the basin is in open air. Several runs are tested and checked for inspection over the total inevitable effects due to deformation of waves by wind, reflection, diffraction, breaking, instability of waves and interference with sensors. The primary waves detected repeatedly at the positions closely located as Fig-2-4 show a good agreement in each other. However, the records of the tertiary wave fluctuates with about 8% of standard deviation. For the frequency, although the motor speeds could be kept constant to within 0.16%, the spectral estimate (2−2) has a width of Δf so that the precision of γ is evaluated to $\Delta\gamma = \Delta f / f_2 \sim 0.017$.

The elements of the mechanically generated waves used in the experiment are shown in Table−2−1.

2.4 Initial Growth of Tertiary Resonant Wave

First of all, we examine whether the tertiary resonant wave $\mathbf{k}_3$ predicted by the theory grows in a basin or not, when we generate a pair of waves $\mathbf{k}_1$ and $\mathbf{k}_2$ mechanically by the wave-makers. An example is shown in Fig−2−6. In this case, $\gamma = 1.793$ and the sensor is located at 45 m from the first wave-maker (nearly mid-point of the basin). In this figure, there appear clearly three line spectra, the lower two lines corresponding to $f_1 = \omega_1 / 2\pi = 1.016$ and $f_2 = \omega_2 / 2\pi = 0.566$ are due to the waves generated by the wave-makers. Remaining one found in higher range is the wave generated by the waves of frequencies f_1 and f_2. The frequency of this component is $f_3 = 1.475$ and it just agrees with the theoretically predicted $2f_1 - f_2 = 1.466$ within the resolution $\Delta f = 0.0098$. This relation holds good in every case of different values of f_1 and f_2. From this Figure, one can see that the resonant wave which is to be a third order quantity in theory exceeds the other second order harmonic components and the amount reaches as 50∼60% of the first order primary wave. This ratio is more than twice as large as those reported in the previous experiments.

In order for reexamining the previous experimental results, we evaluate the initial growth rate of resonant waves and its dependence upon the frequency ratio γ of the primary waves. As explained in §1.3, the initial growth rate G is connected to observable quantities such that

$$A_3 / d\,(A_1 k_1)^2 (A_2 k_2) = G(\gamma,\ \theta)\ |\sin\delta k d / \delta k d|, \qquad (2-3-1)$$

$$\frac{2\delta k}{k_3} = -\left(\frac{4}{2\gamma_0 - 1} - \frac{8\gamma_0^3}{4\gamma_0^4 + 1}\right)(\gamma - \gamma_0). \qquad (2-3-2)$$

Where, d is the fetch of interaction, δk is the detuning wave-number of primary waves and θ is the angle of intersection.

The initial growth rate G was evaluated 0.442 when $\theta = \pi / 2$ and $\gamma_0 = 1.736$ (the value of G is nearly constant with γ around γ_0). In Fig-2-7, the values of the left-hand side of (2-3-1) calculated from the measurement data at the location in Fig-2-4 (Case I) are shown against γ. In this case, the wave gauges are located near to the wave-maker 1 to obtain the initial growth data. The solid curve is drawn by the right-hand side of (2-3) fitted by inspection with G and γ as parameters. From Fig-2-7, it is estimated that G=0.50 and $\gamma_0 = 1.79$. A comparison with McGoldrick's result is shown in Table-2-2. In this initial stage, the results of γ_0 agree fairy well and are somewhat greater than that of the theory. This fact will be partly explained by the concept of NON-LINEAR RESONANCE CONDITION introduced in Chapter 3. The value G in this experiment lies between the value of their experiment and the classical theory.

Also by means of this location of wave gauges (these six gauges are tightly attached to a stainless steel bar with the mutual distances of 0.45m, 1.05m, 1.20m, 0.60m, 0.30m as consisting a linear array), the determination of the direction of tertiary wave which has not executed in the previous papers is examined. By the theory due to Longuet-Higgins(1962), the angle of tertiary wave to the primary first wave is predicted 9.24 degrees for the case of exact resonance.

Defining the mutual distance between wave gauges D_{12} and the relative angle to the wave α shown as in Fig-2-8, the phase difference ϕ_{12} of the wave for D_{12} is written as

$$\phi_{12} = k D_{12} \sin \alpha \quad , \qquad (2-4)$$

where, k is the wave-number concerned. Otherwise, phase difference can be calculated from the data obtained at two wave gauges by their CROSS SPECTRUM. If co-spectrum and quadrature-spectrum are expressed as C_{12} and Q_{12}, ϕ_{12} is correlated by them as

$$\phi_{12} = \tan^{-1} (Q_{12} / C_{12}) \ . \qquad (2-5)$$

In Fig-2-9, we show the coherence among the data measured with the wave gauges 1 and 3. Although there appears some broadening around the second primary wave, the coherence is almost nearly unity at around the three wave frequencies considered here. Fig-2-10 shows the phase spectrum of this data. Choosing every pair of gauges from six, the phase $\delta=\phi/\pi$ of three waves f_1, f_2, f_3 is described against D_{lm} (D_{lm} ($l,m=1,2,\cdots,6$) is distributed not to be equal for every pair of the gauges) in Fig−2−11(a),(b),(c) respectively. Using the data $k_1=3.993$, $k_2=1.291$, $k_3=8.188$ in the formulae

$$\delta_n=(k_n D_{l,m}/\pi)\sin\alpha_n, \qquad (2-6)$$

where $n=1,2,3$ and $l,m=1,2,\cdots\cdots,6$,

we can determine α_1, α_2, α_3 from the tangent of each plot. The straight lines in Fig−2−11 are obtained by means of the least square method. By these Figures, we can estimate that $\alpha_1=1.09$, $\alpha_2=73.40$ and $\alpha_3=-7.85$degrees so that the direction of tertiary wave from the first primary wave is $\alpha_3-\alpha_1=-8.94$degrees, while the theoretical prediction in this case is −9.19 degrees. We can recognize that the agreement of both values is satisfactory.

2.5 Long Term Evolution of Tertiary Resonant Wave

In this section, we investigate the long term behavior of the tertiary resonant wave. In order to perform this task, wave gauges are arranged as shown in Fig−2−5 (Case II). Six wave gauges are set at the distance from the first wave-maker of 26.56 m, 35.96 m, 41.15 m, 45.36 m, 50.95 m and 61.15 m along the direction of tertiary waves. They are the very longer fetches than those of the Case I and those of the previous works (their maximum span of observation is about 15 m after transforming the size to the present experiments).

The results are explained in the following. In Fig−2−12 through Fig−2−18, the amplitudes of tertiary waves are plotted against the distances along the direction of propagation.

For describing the observational results clearly, we explain them corresponding to the experimental conditions in order:

1) $\gamma\sim1.72$ (nearly resonant), A_1 and A_2 ($\sim$2.5cm) are both small. {Fig−2−12}

In this case, the growth of resonant waves are nearly straight. The broken line shows the theory of Longuet-Higgins(1962) (equation (2−3)). The long fetch behavior can be explained in this case qualitatively by the classical theory.

2) $\gamma \sim 1.72$ (nearly resonant) , A_1 ($\sim$4cm) is larger than the case ①. {Fig－２－１３}
While A_2 ($\sim$2.5cm) is as same as the case ①, resonance does not strongly occur and the amplitude of tertiary wave is in every point small. The curve represents the quasi-stationary solution given by (３－８) by means of the Zakharov theory.

3) $\gamma \sim 1.79$ (off resonant) , A_1 is small and A_2 ($\sim$ 5cm) is moderate. {Fig－２－１４}
In this case, A_3 is nearly constant (slowly varying) thorughout the fetch where the measurements are made. The manner of variations looks almost parallel and the values are found larger as A_1 increases from 1.80 to 2.84. In the last case (A_1 is the largest) , the values of A_3 amounts to about 1.5 cm. The appreciable values of resonant waves are observed in the first time in such a off resonant cases.

4) $\gamma \sim 1.79$ (off resonant) , A_1 is larger than the case 3) while A_2 is small. {Fig－２－１５}
This is rather curious result. Although the condition is so far from the case 1), the growth of A_3 is clearly straight. The broken line in this figure is the theoretical one like the item 1) (omitting the detuning factor) . The dashed-and-dotted line is determined by the least square fitting. Looking at the discrepancy between both lines, it suggests that in this case, a sort of non-linear resonance condition including the amplitude dependence to the wave velocity would hold and it suppresses the free evolution of tertiary wave.

5) $\gamma \sim 1.79$ (off resonant) , A_1 is larger than case 3). {Fig－２－１６}
A_3 clearly decreases as the fetch increases and diminish to zero (recurrence phenomena) instead that the asymptotic steady states take place in a longer fetch.

6) $\gamma \sim 1.82$ is larger, A_1 and A_2 are both large. {Fig－２－１７}
In this case, it is characteristic under this condition that the magnitudes of A_3 decrease initially as the fetch increases and then grow up once again. Subsequently resonant waves repeat the same process. However this is not sure in the present experiment because the length of the basin is not enough long to pursuit this character. This tendency appears the faster (at the shorter fetch) with the larger A_1.

7) The largest wave obtained in this experiment is shown in Fig－2－18. In this experiment, tertiary resonant waves has never exceeded 2.5 cm in amplitude (5 cm in wave height). This limitation may depend upon the wave-steepness of the primary waves used in this experiment. Local breaking of waves is apt to arise particularly in such a composed wave system that mechanically and spontaneously generated waves consist of a comparatively wide spread frequency components. These local breakers possibly prevent the resonance mechanisms from being sufficiently enhanced.

8) In the case of γ far from γ_0, say $\gamma < 1.6$ or $\gamma > 2.0$, it is verified that no wave is generated at all.

In general, the straight resonant growth is seriously dependent on the conditions among the frequencies and amplitudes of primary waves. On the contrary, the recursive resonant growth occurs in somewhat soft conditions whereas the maximum values of them are comparative to the former. The decreasing of amplitudes of tertiary waves at the longer fetch rather reveals that the strong interaction takes place even in this region, otherwise the resonant waves which are once generated at shorter fetch would travel to the outer region without decaying their amplitude at all.

The tertiary resonant waves generated by mutual interaction of primary waves can be observed by the naked eye in this experiment. Since the wave velocity of tertiary wave is much less than the primary waves, it can be left in the basin after stopping the wave-makers and passing the primary waves away to the absorbing beaches. This fact is another confirmation that these tertiary waves are free waves in accordance with the theory. Three photographs on the experiment of the generated resonant wave are shown in Fig－2－19. The direct photographic observation of deep-water gravity wave interaction had not been known in the past. From the picture of Fig－2－19 (c), wavelength of the tertiary wave taken in the photo is measured as 72.5 cm. While the theoretical length is 71.7 cm.

CHAPTER 3 NUMERICAL SOLUTION OF ZAKHAROV EQUATION

3．1 Foreword

The non-linear theory described in §1．5 gives an integro-differential equation which governs the slow variations of first order amplitude and phase components among multiple directional waves. This type of equation was first derived by Zakharov(1968), and is called the ZAKHAROV EQUATION. In general, it is difficult even to obtain the solution of this equation by numerical method, not to mention to solve it analytically. So, the Zakharov equation has never been applied except for the stability problems of monochromatic wave train.

In this Chapter, we deal with this equation in the most important case of three waves mutual interaction by regarding it as a system of ordinary differential equations. At first, a simple approximate solution to this system of equations is derived analytically assuming that the energy transfer among waves is not so large. This solution lends itself to consider the resonance condition with the amplitude effect taking into account. In the next place, the measurements at shorter fetches given by McGoldrick et. al. (1966) is successfully compared with this theory. A simple and clear evaluation of the limiting wave height of resonant waves is also put forward in terms of the first primary wave amplitude. The result is confirmed numerically by the repeated execution of long-time numerical integration of this system of equations. Through this calculation, recurrence properties which are found and described to some extent in Chapter2 are reproduced.

The comparison of the results are made with experiments described in Chapter2, and the comprehensive discussion on the resonant interaction phenomena are yielded in Chapter4. At the last section of this Chapter, a related problem on instability prorerties of a quasi-monochromatic wave train are treated by the same method. The relation of this equation with Hasselmann's energy flux equation among continuous spectral component is interpreted in AppendixⅢ. The relation with Non-linear Schroedinger equation is also explained in AppendixⅣ.

3．2 Numerical Experiment

The fundamental integro-differential equation has the form

$$i\frac{\partial B(\mathbf{k},t)}{\partial t}=\iiint_{-\infty}^{\infty}d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 T(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$$

$$B^*(\mathbf{k}_1,t)B(\mathbf{k}_2,t)B(\mathbf{k}_3,t)\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)$$

$$\exp\{i(\omega+\omega_1-\omega_2-\omega_3)t\}. \qquad (3-1)$$

This is conceptually equivalent to (1 − 3 4). In this expression, the simbol δ is Dirac delta-function. The explicit form of the kernel T is presented in Appendix V. Using the quantity B, surface elevation η is expressed as

$$\eta(x,t)=(2\pi)^{-1}\int_{-\infty}^{\infty}(k/2\omega)^{-1}d\mathbf{k}B(\mathbf{k},t)\,\mathrm{expi}(\mathbf{k}\cdot x-\omega t).$$

$$(3-2)$$

Pulling out from (3 − 1) the three components discussed in Chapter 2, it is transformed into ordinary differential equations as

$$i\frac{dB_1}{dt}=[T_{1111}B_1B_1^*+\widetilde{T}_{1221}B_2B_2^*+\widetilde{T}_{1331}B_3B_3^*]B_1+$$

$$\widetilde{T}_{1123}e^{i\Delta\omega_{1123}t}B_1^*B_2B_3, \qquad (3-3-1)$$

$$i\frac{dB_2}{dt}=[\widetilde{T}_{2112}B_1B_1^*+T_{2222}B_2B_2^*+\widetilde{T}_{2332}B_3B_3^*]B_2+$$

$$T_{2311}e^{i\Delta\omega_{2311}t}B_3^*B_1B_1 \qquad (3-3-2)$$

and

$$i\frac{dB_3}{dt}=[\widetilde{T}_{3113}B_1B_1^*+\widetilde{T}_{3223}B_2B_2^*+T_{3333}B_3B_3^*]B_3+$$

$$T_{3211}e^{i\Delta\omega_{3211}t}B_2^*B_1B_1. \qquad (3-3-3)$$

These are actually the six degree non-linear equations with respect to the real and imaginary parts of B.

Where, T_{1234} denotes $T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ and conventional notations $\widetilde{T}_{1234} = T_{1234} + T_{1243}$, $\Delta\omega_{1234} = \omega_1 + \omega_2 - \omega_3 - \omega_4$ are used. It is confirmed that this discretized approximation is self-consistent and the other components play no role at least in the first order, if they does not exist a priori. The first terms in the right-hand sides of (3 − 3 − 1) ~ (3 − 3 − 3) represent the phase velocity effect in tertiary wave interaction which is briefly interpreted in AppendixVI.

Before solving (3 − 3 − 1) ~ (3 − 3 − 3), we discuss about the conservation laws of this system.

Taking notice on the magnitude of B, the symmetrical property of the equations leads that

$$2\widetilde{T}_{1123}^{-1}|B_1|^2 + T_{2311}^{-1}|B_2|^2 + T_{3211}^{-1}|B_3|^2 = \text{const}$$

and (3 − 4 − 1)

$$T_{2311}^{-1}|B_2|^2 - T_{3211}^{-1}|B_3|^2 = \text{const} \quad . \qquad (3-4-2)$$

From the expressions (3 − 3 − 1) ~ (3 − 4 − 2), one can immediately notice for the energy transfer among these three waves that the first primary wave B_1 shears its energy to B_2 and B_3 for growing them, that is, the energy flows from B_1 toward B_2 and B_3, or vice versa. The first primary wave B_1 plays the most fundamental role in this interaction and unlike it, the role of the second primary wave B_2 is subsidary.

Considering that the complex amplitude B has a relation with the actual wave amplitude A as

$$|B(\mathbf{k})| = \pi\left(\frac{2\omega}{\mathbf{k}}\right)^{\frac{1}{2}} A(\mathbf{k}), \qquad (3-5)$$

it leads to

$$|B(\mathbf{k})|^2 = \pi^2\left(\frac{2g}{\omega}\right) A(\mathbf{k})^2 \quad . \qquad (3-6)$$

Because A^2 is proportional to the energy of waves, $|B(\mathbf{k})|^2$ means the wave action (see Leibovich et.al.(1974) or Phillips(1977)) in this system. Conservation laws are interpreted in more details in AppendixVII.

In the next step, we examine an approximate analytical solution of equation (3 − 3 − 1) ～ (3 − 3 − 3). In this approximation, we assume that the amplitude of resonant wave is much less than those of the primary waves. By use of this assumption, we neglect the terms containing B_3 in (3 − 3 − 1) ～ (3 − 3 − 3). In this manner, the amplitudes of the primary waves are regarded as constants so that the quantities in [] of (3 − 3 − 1) ～ (3 − 3 − 3) should be also constants. They are denoted by θ_1, θ_2 and θ_3 ($\Delta\omega = 0$ is set without loss of generality), that is,

$$\theta_1 = [T_{1111}B_1B_1^* + \tilde{T}_{1221}B_2B_2^*],$$

$$\theta_2 = [\tilde{T}_{2112}B_1B_1^* + T_{2222}B_2B_2^*]$$

and

$$\theta_3 = [\tilde{T}_{3113}B_1B_1^* + \tilde{T}_{3223}B_2B_2^*].$$

Representing $B_n(t) = b_n(t)\exp i\chi_n(t)$, (n=1, 2, 3) under the constraint of b_n, χ_n being real functions, we get from (3 − 3 − 1) and (3 − 3 − 2) that $b_1(t) = b_{10}$, $b_2(t) = b_{20}$, $\chi_1(t) = -\theta_1 t$ and $\chi_2(t) = -\theta_2 t + \pi/2$. Using them to the last equation (3 − 3 − 3), it reduces to

$$\frac{db_3}{dt} = T_{3211}b_{10}^2 b_{20}\cos\{(2\theta_1 - \theta_2)t + \chi_3\} \tag{3-7-1}$$

and

$$\frac{d\chi_3}{dt} = -\theta_3 - T_{3211}b_{10}^2 b_{20} b_3^{-1}\sin\{(2\theta_1 - \theta_2)t + \chi_3\} \tag{3-7-2}$$

which are non-linear equations with respect to b_3 and χ_3. Considering the initial condotion $b_3 = 0$, $\chi_3 = 0$, at $t = 0$, we introduce an undetermined constant β as $\chi_3 = -\beta t$ and integrate (3 − 7 − 1). The result is

$$b_3 = \{K/(2\theta_1 - \theta_2 - \beta)\}\sin(2\theta_1 - \theta_2 - \beta)t$$

where $K = T_{3211}b_{10}^2 b_{20}$. Substituting χ_3 and b_3 into (3 − 7 − 2), we can determine β as follows

$$-\beta = -\theta_3 - (2\theta_1 - \theta_2 - \beta), \text{ that is, } \beta = \theta_1 - \tfrac{1}{2}\theta_2 + \tfrac{1}{2}\theta_3 .$$

Thus,

$$\chi_3 = -\left(\theta_1 - \tfrac{1}{2}\theta_2 + \tfrac{1}{2}\theta_3\right) t .$$

Accordingly, time variation of b_3 can be decided as

$$b_3 = \{K / (\theta_1 - \tfrac{1}{2}\theta_2 - \tfrac{1}{2}\theta_3)\} \sin(\theta_1 - \tfrac{1}{2}\theta_2 - \tfrac{1}{2}\theta_3)\ t \quad . \tag{3-8}$$

It is easy to verify that this pair of solutions χ_3 and b_3 satisfies the equation (3 − 7 − 1) and (3 − 7 − 2) exactly. In the initial stage of evolution, the solution (3 − 8) reduces to $b_3 = K\,t$, which would be equivqlent to the classical result (Longuet-Higgins(1962)) . In order to verify whether the theory of Zakharov equation to be applicable to the phenomena or not, we now compare (3 − 8) with the experimental results given by McGoldrick et. al.(1966)as the initial growth of tertiary waves. In accordance with their experimental parameters, we rewrite (3 − 8) as

$$a_3 = (4\pi^2 T_{3211}) \left(\frac{\omega_3}{\omega_1}\right) \left(\frac{\omega_3}{\omega_2}\right)^{1/2} a_1^2 a_2 d \ , \tag{3-9}$$

where d is the fetch of interaction. We adopt concrete values on the basis of their experiment as $a_1 = 0.32$ cm, $a_2 = 0.895$ cm, $\omega_1 = 16.87\ \mathrm{sec}^{-1}$, $\omega_2 = 9.65\ \mathrm{sec}^{-1}$ and $\omega_3 = 24.0\ \mathrm{sec}^{-1}$. Thus, we can calculate the amplitude of tertiary waves against fetch d by evaluating the coupling coefficient $T_{3211} = 40.964$ from the Zakharov theory. The results under the condition mentioned above, together with the case that $a_2 = 0.45$ cm (one-half of the former) with the symbols ○ and △ respectively are drawn in Fig− 3 − 1 . Their data on two series of experiment show good agreement with the Zakharov theory. In this paper, we call (3 − 8) the QUASI-STATIONARY solution of the equations (3 − 3 − 1) ～ (3 − 3 − 3) .

Using the quasi-stationary solution verified to be valid immediately before, we consider the NON-LINEAR RESONANCE CONDITION, that is, the dependence of $\gamma_\emptyset$ upon the amplitudes of primary waves. Slight extension to the solution (3 − 8) when $\Delta\omega \neq 0$ yields the modification of its argument as $\theta_1 - \frac{1}{2}\theta_2 - \frac{1}{2}\theta_3 + \frac{1}{2}\Delta\omega$. Therefore, by this approximation the non-linear resonance condition is expressed as

$$\Delta\omega + 2\delta = 0, \qquad (3-10)$$

where δ is given such that

$$2\delta = [2T_{1111} - \widetilde{T}_{2112} - \widetilde{T}_{3113}]\ B_1 B_1^{*} +$$
$$[2\widetilde{T}_{1221} - T_{2222} - \widetilde{T}_{3223}]\ B_2 B_2^{*}. \qquad (3-11)$$

It is obvious that for the linear resonance condition, (3-10) merely reduces to $\Delta\omega = 0$ and $\gamma_0 = 1.7357\cdots$. For evaluating the small correction γ', we assume that the non-linear resonance condition γ_m is expressed by $\gamma_m = \gamma_0 + \gamma'$ and approximate (3-10) up to the first order of γ'. The result is

$$\gamma' = \frac{-(8\gamma_0^3 - 12\gamma_0^2 + 6\gamma_0 - 1)}{2(6\gamma_0^3 - 12\gamma_0^2 + 6\gamma_0 - 1)}(2\delta). \qquad (3-12)$$

(3-12) together with (3-11) represents a correction of the resonance condition by finite amplitudes of primary waves.

If we apply the Zakharov's coefficients, the non-dimensional formula is derived that

$$\gamma' = 1.66055(a_1 k_1)^2 - 2.74992(a_2 k_2)^2. \qquad (3-13)$$

An example for the case calculated in § 3. 3, that $a_1 = 4.7$ cm, $a_2 = 5$ cm, $\lambda_1 = 1.66$ m and $\lambda_2 = 4.99$ m leads to $\gamma' = 0.051838\cdots$. Thus we obtain the value $\gamma_m = \gamma_0 + \gamma' = 1.788$ which agrees fairly well with the value $\gamma = 1.800$ adopted in the caluculation.

In the following section, we mention how tertiary wave amplitude A_3 is correlated with the changes of the amplitudes A_1, A_2 and the frequencies ω_1, ω_2 of the primary waves. The details of the numerical procedur are referred to Tomita(1987).

3. 3 Behavior of the Tertiary Resonant Waves

We can integrate the equation (3-3-1) ~ (3-3-3) numerically under the condition that amplitudes A_1 and A_2 are given as concrete values in the experiment. The amplitude A_3 is assumed to be zero initially. The phase difference between the primary waves has no influence upon the results. Corresponding to various values of A_1 and A_2, long time variations of three waves are shown in Fig－3－2, and Fig－3－3. It is shown that the energy exchange occurs among the three waves and the amplitudes of waves vary periodically (not always sinusoidal) and never reach any equillibrium (this problem is discussed in more detail by the analytical investigation of these equations at the latter part of this paper). Fig－3－2 corresponds to the case $\gamma=1.735$ (near resonant). Initial values of A_1 are prescribed (a) 1 cm (b) 2 cm (c) 3 cm and (d) 4 cm in order, while A_2 is fixed as 5 cm. The growth of A_3 is apparently limited. The straight line A_3' in Fig－3－2 (b) is the solution given by Longuet-Higgins(1962). It means that the solution of Zakharov equation reduces close to the classical one in the initial stage $t \ll 1$ as mentioned at the previous section.

On the contrary, when $\gamma=1.800$ (off resonant) A_3 grows to some extent according to the increase of A_1 (see Fig－3－3 (a) ～ (e)). Initial values of A_1 are prescribed (a) 2 cm (b) 3 cm (c) 4 cm (d) 4.6 cm and (e) 5 cm, while A_2 is fixed as 5 cm. In Fig－3－3 (d), the amplitude A_3 temporarily exceeds the first order quantity A_1. If we set that $A_1=5$ cm initially, the growth of A_3 rather reduces.

In the second place, drawing our attention to the nature that A_3 reaches their maximum values in finite durations in any cases, we investigate the values of the maximum A_{3max} against A_1 with γ as a parameter. A result when A_2 is fixed as 5 cm, is shown in Fig－3－4. In Fig－3－4, the parameter R which is square of γ (the exact resonance ratio $\gamma=1.736$ discussed in Chapter 2 corresponds to $R=3.014$) is used. The value of γ is also shown in Fig－3－4. When $R>3.0$, each solution A_{3max} corresponding to different values of R has sharp peak A^M_{3max} in the vicinity of each value A_{1R} without regard to R. The fact is also noticed that in the case $R>4.0$, each solution A_{3max} as a function of A_1, is nearly identical without respect to R. It is obvious from the mathematical point of view. The reason is understood that the Zakharov coefficients T_{abcd} (a, b, c, d=1, 2, 3) does not vary so much with R. Whereas, physically speaking, it is not so obvious. We merely point out that A^M_{3max} exceeds the highest limit of gravity wave, hence the formulation of the theory up to the third order of wave steepness would be insufficient under this condition.

A^M_{3max} is a quantity which is characteristic to express the intensity of resonant wave interactions. Using the results discussed in AppendixⅦ, we have a criterion about the limit of maximum amplitude of tertiary wave A^M_{3max} that it depends only on A_1 (not on A_2) linearly such that

$$A^{M}_{3max} = 0.844 A_1 \quad . \qquad (3-14)$$

The maximum value A_{3max} of tertiary resonant waves could grow to the extent of 84% of the first primary wave which generates it. In the case $R < 2.9$, the curves run close to the abscissa, that is, the small part of energy can be transfered. For the case of amplitude of the second wave A_2 is 10 cm, the maximum of tertiary wave A_{3max} is shown in Fig−3−5. The broken line in Fig−3−5 is (3−14) which passes through the each maximum of A_{3max}, in this paper, we express it as A^{M}_{3max}. We could not examine this formula (3−14) directly, because it is difficult to generate sufficiently large amplitude wave which has non-deformed, non-breaking crest lines of constant height with the wave-makers used in this experiment.

Finally, we execute several numerical integration by arranging the initial values of the amplitudes of two primary waves as real value recorded in the experiment. Examples are displayed in Fig−3−6∼Fig−3−7. As is explained in Chapter 2, two primary waves could not directly be compared with the theoretical ones because their amplitude are affected more intensely by inevitable effect of wave diffraction than that of interactions. The plotting is done only for tertiary waves. In the theories, phenomena are assumed to be uniform in space and vary with time. On the contrary, for the experiment, we make up the stationary state in a basin and detect the spatial variation of wave amplitudes with several wave gauges. By this reason, the wave amplitude of tertiary waves are drawn against spatial fetch d. Fig−3−6 for $R = 2.97$ ($\gamma = 1.72$), Fig−3−7 for $R = 3.21$ ($\gamma = 1.79$) show the data with the Zakharov's theoretical values. Between the both examples, the manner of variation of A_3 are somewhat different, nevertheless the agreement of the data with the theory are fairy well. However, as is seen at the last example, Fig−3−7 (c), experimental data do not amount so large as half as that of the Zakharov theory when the wave height is extremely large.

In order to show the applicability of the theoretical results to the scale of actual sea, we present a summary concerning the similarity problem in Appendix Ⅷ. Analytical properties of interaction equations (3−3−1) ∼ (3−3−3) are briefly investigated in Appendix Ⅸ with the proof for existing of the steady state asymptotic solution corresponding to the specific amplitudes of first and second primary waves.

3．4 Instability Properties of a Wave Train

We must mention first of all, the famous study by Benjamin & Feir (1967) with regard to this problem. By their theory, a finite amplitude deep-water wave train is unstable to the small subharmonic disturbances whose components have a pair of side-bands of ω, say $\omega + \Delta\omega$ and $\omega - \Delta\omega$. They restricted themselves to one-dimensional problem that disturbing waves advance in the same direction as the primary wave. Recently, Crawford, Lake, Saffman & Yuen(1981) by means of the Zakharov equation, MacLean(1982)by use of the exact Eulerian equations calculated the domain of stability to two-dimensional perturbations in the framework of linear instability theory. Su, Bergin, Marler & Myrick(1982), Su (1982) also carried out the experimental studies in a very long wave flume in open air and indicated the importance of the two-dimensional perturbations to the stability of steep gravity waves. Observations on a modulational characteristics of wind waves were conducted by Mase et.al. (1985) and Donelan(1987)in actual sea. The former authors successfully compared their observational data with the computational results from the Zakharov equation. We should also refer the studies on a instability of non-linear standing water waves elaborated by Okamura(1984, 1985) using the Zakharov equation. In this section, we utilize (3－3－1)～(3－3－3) as they are, to investigate a monochromatic wave which is exerted by two-dimensional small perturbations.

To this type of problems, B_1 is recognized as a primary wave and B_2, B_3 as a pair of side-band perturbations advancing in the different directions. In Fig－3－8 one can see an example of the long time behavior of each components A_1, A_2 and A_3. The perturbation components rise up spike-wise intermittently in all the cases to be examined. In these calculations, the magnitudes of small perturbations A_2 and A_3 are initially set 10^{-6} times as large as that of primary wave A_1. The height and the recurring period of spikes are intrinsically dependent on the wavelengths and directions of two perturbational component waves. Thus we examine the waves of wave-numbers $\mathbf{k}_1 = K_1(1, 0)$, $\mathbf{k}_2 = K_1(1+p, q)$ and $\mathbf{k}_3 = K_1(1-p, -q)$ in the regions $0 < p < 1.2$, $0 < q < 0.5$. The vector $K_1(p, q)$ is taken as a modulational wave-number. The instability criterion is defined that the side-band amplitudes exceeds 10^{-4} times as large as that of the primary wave. The domain of instability calculated by the Zakharov equation are shown in Fig－3－9. Small circles mean the unstable couples (p, q). The stable results are not illustrated in the Figure, e.g., the wave is stable in the region except where is filled by the grid of the small circles.

Solid curves drawn in the same Figures represent the boundary of the domains of instability by means of linear stability theory after MacLean (1982). From the numerical experiment executed in this time, side-band components rise up abruptly at the outer side boundary of the wave-number space.

The stability property of a wave train can be investigated in the same manner as resonant problem, conversely, the stability property has not been sufficiently taken into accounts in studying the resonant interactions. In this study, Zakharov equation was discretized into the most important three wave components. However, from the stand point mentioned in this section, the affection on the resonant interaction properties by other components must be investigated. There would be certain contributions through the instability and phase speed effect exerted on primary waves by other components neglected in this paper. To a further step, the computation with a great many components are desirable for directly simulating the actual ocean wave spectra, possibly by use of super computer. It will be an issue to be treated more comprehensively in the future studies.

CHAPTER 4 CHARACTERISTICS OF THE RESONANT WAVE INTERACTION

4.1 Outline of the Preceding Chapters

In the previous chapter, we explained the classical theory due to Longuet-Higgins(1962)in § 1.3 and the more comprehensive theoretical approach to long term evolution of resonant interactions in §§ 1.4 and 1.5. The experimental results shown in Chapter 2 revealed that the classical theory is insufficient to describe the observational results. In Chapter 3, the Zakharov equation which is applicable to long term variation of non-linear waves was numerically integrated and the experimental data were partly confirmed to agree with the theory in several examples. In this Chapter, the experimental data are compared with these theories in more entire point of view. For this purpose, we summarize the facts obtained in the preceding Chapters as follows:

1) The existence of tertiary wave generated by resonant interaction is verified experimentally. The tertiary wave which grows up to 62% as large as the first primary wave is detected when $\gamma = 1.79$ and $d = 45.36$m.

2) In short fetches, the growth rate of resonant waves is somewhat smaller than that measured by McGoldrick et.al.(1966) and greater than theoretical value given by Longuet-Higgins(1962) by 18%. Tertiary wave growth takes place most strongly at the value of $\gamma = 1.78$ which is slightly different from the exact resonance condition $\gamma_{\emptyset} = 1.736$. This is partially interpreted by the non-linear correction of the resonance condition.

3) The measurements done by McGoldrick et.al. in the short fetch are completely accounted by the Zakharov theory.

4) The direction of propagation of generated tertiary resonant waves are determined about 9 degrees which is identical with the theory.

5) In longer fetches, tertiary wave grows up to their maximums and diminishes its amplitude. The fetch lengths for the recurrences depend upon the amplitudes of primary waves A_1 and A_2. This fact can not be explained by classical theory (expressed in (1 − 9)).

6) In general, the behavior of tertiary waves is affected not only by the frequency ratio γ which is related to resonant condition, but also by

the amplitudes of primary waves A_1 and A_2.

7) By comparison of the Zakharov equation with the observational data, we can see that the theory explains the evolution of tertiary waves at the case of small steepness of each waves. However, the discrepancies become large with increase of the wave steepness.

8) An approximate analytical solution of Zakharov equation (3 − 8) is proposed. The observational results are explained by this solution when energy transfer among waves is comparatively weak.

9) By solving the Zakharov equation repeatedly, the maximum values A_{3max} realized by tertiary waves are determined against A_1 with γ as a parameter.

As the quantitiy A_{3max} is suitable to discuss about the entire characteristics of resonant wave interactions, we rearrange the data to be compared with theories through this concept.

4. 2 Comparison with Classical Theory

The recurrence properties of tertiary waves are explained even by the theory of Longuet-Higgins(1962) if we recognize them as an effect of detuning of the frequency ratio γ to its prescribed value γ_0. According to this theory, the maximum value A_{3max} to be realized is yielded by (1 − 14). In Fig− 4 − 1, we take the theoretical values to the abscissa and those of experimental values as the ordinate and plot the points in the graph of dispersion. If theory and experiment agree with each other, the points should be distributed on a line drawn in Fig− 4 − 1. The result scatters to a large extent. This means that the theory can not explain the experimental results. Taking into consideration that there are results for many cases of γ in Fig− 4 − 1, we classify them into three main categories. The symbol △ corresponds to the case $\gamma \sim 1.72$ (nearly resonant case). In this case, all the data run close to the x-axis. It means that resonant waves do not so evolved as the classical theory predicts. Symbols □ correspond the case $\gamma \sim 1.79$ (not so close to the resonant case). The data are seen to wind themselves around the line and comparatively near to it. The dispersion seems not to be random. The points are distributed higher for small A_{3max} and lower for large A_{3max} than the solid line. The case $\gamma > 1.82$ is plotted by the symbol ○. In this case, all the points are plotted above the line, in other words, the measured values are always larger than that of the theory. The reasons why the theory and experiment are not identical in general is suggested as follows:

Firstly, the velocity of tertiary wave changes by the influence of non-linear amplitude dispersion. Velocity becomes slightly larger with increase of the wave amplitude. As a result, resonance system of wave-wave interactions turns out of tune. On the contrary, for the case that resonance condition is not so closely satisfied, exact resonance can be preserved by a slight-detuning to compensate for amplitude dispersion inferred by Phillips(1977). Moreover, primary waves shed their energy to the other waves to intensify the growth of resonant waves. These effects were not considered in this classical theory.

4．3 Comparison with Zakharov's Theory

In order to clarify the effect of the primary wave amplitude A_1 to the growth of tertiary resonant wave quantitatively, we examine the dependence of the maximum amplitude of tertiary wave A_{3max} on A_1 in the sequel. The experimental values of A_{3max} are plotted this time against A_1 in Fig－4－2. There seems no clear tendency in Fig－4－2. We arrange these data in the following manner. As same as the prvious section, the set of data is classified by the magnitudes of γ.

The case $\gamma \sim 1.72$ is shown in Fig－4－3. Theoretical curve calculated by means of Zakharov equation is also shown in Fig－4－3. Taking various noise described in Chapter 2 into considerations, the agreement of the theoretical prediction with the acquired data is fairy well in this case. Fig－4－4 shows for the value $\gamma \sim 1.79$. It is very characteristic in this figure that the theoretical curve expresses the existence of strong resonance in the vicinity of $A_1 \sim 4.0$ cm. The measured data agree well with this characteristics. Although the sharp peak for A_{3max} is not observed experimentally, the discrepancy might be caused by that the waves made with wave-makers are not perfectly monochromatic, so the critical condition demanded by the theory for the peaks would not be realized. On the other hand, higher order effects which are not considered in the theory might have an influence under such a subtle condition. In Fig－4－5, the case $\gamma > 1.82$ is totally plotted. The data are somewhat widely scattered in this Figure, however considering the instability property of waves at the large amplitude, it is concluded that the entire behaviors obtained in the experiment could be explained by the theory of Zakharov equation.

4. 4 Discussion

It is confirmed that the long term evolution of tertiary resonant waves are not explained by the classical theory. On the other hand, by the comparisons of the experiment with the theory of Zakharov we can conclude that this theory is applicable to this sort of phenomena. It could predict the evolution of tertiary resonant waves under the conditions that the wave steepness $H/L<0.05$ (the reproducible experiment was conducted to the wave whose steepness is less than 0.05) and the frequency ratio $1.58<\gamma<1.90$. It is the point left as an open question when one applies this sort of equations derived by the singular perturbation method. Although these criteria are not determined directly by the experiment, they are discussed briefly in AppendixⅦ.

After all we summarize the over all properties of the generation of tertiary resonant wave by perpendicularly intersecting two primary waves as follows:

1) In general, resonant waves show a spatial (temporal) recurrence (periodicity). The non-linear resonant wave interaction phenomena are interpreted by a third order slowly varying theory using the Zakharov equation.

2) Growth rate G for short term development is proportional to the square of the first primary wave amplitude A_1. Classical theory is valid only for this region.

3) The resonance takes place most strongly at the "off-resonance" condition $\gamma\sim1.79$ in terms of the linear dispersion relation. Introduction of the concept "non-linear resonance" is necessary.

4) To the values of γ less than γ_0, there exists no strong resonance and A_{3max} approaches to a small constant value without respect to A_1.

5) For the cases $\gamma<1.6$ and $\gamma>2.2$, tertiary wave does not appear at all in the experiment.

There were several reports including Snodgrass et.al.(1966) who pointed out the importance of wave-wave interactions in a seaway. Mollo-Christensen & Ramamonjiarisoa(1978, 1982) proposed a new model for ocean waves described by the presence of wave groups in a random wave field. Chereskin & Mollo-Christensen(1985) conducted an experimental study about the amplitude and phase modulation of a one-dimensional wave flume. The coherency of narrow-band weakly non-linear one-dimensional wave system is pointed out by their papers. If such a coherent property is predominant in the ocean waves, resonant interaction would take place more intensely than considered in a model of random wave field. Recently, Sand(1988)reported the topics in the problems of wave forces as a environmental conditions to ocean structures. In the field of research concerning the mooring of off-shore floating structures, for example, investigations into non-linear properties of sea waves will play an essential role in the near future.

The present author(1988b, c) also investigated the wave group characteristics by use of the data obtained with wave buoy at the North Pacific Ocean during 1983～1984. It might be a manifestation of non-linear modulation property of wind waves indicated by Mase et.al.(1985) or Li et.al.(1987). In the present time, observational wave data are not enough to make clear the mechanisms of this sort of unsteady non-linear processes in sea waves. Application of more exhaustive analysis techniques such as INSTANTANEOUS SPECTRUM proposed by Bendat & Piersol(1967) and/or INVERSE SCATTERING METHOD founded by Zakharov & Shabat(1972) and interpreted by Sobey & Colman(1982, 1983) in the context of sea waves seem to be necessary.

Although it is a future problem that the investigations are executed for the more general cases, four wave mutual interactions etc., the co-operative method of study within theory, calculation, experiment and obserbation is indispensable for such a non-linear problem.

ACKNOWLEDGEMENTS

To complete this paper, the author owes to many peoples. The first debt is to Dr.J.Iwayanagi, the former director of Physical Engineering Division of Ship Research Institute(S.R.I.)and Prof.Y. Yamanouchi of Nihon Univ.(former director general of S.R.I.), who had given me an occasion to enter this field of research. I also would like to thank my seniors in this field, Prof.Y.Nagata of the University of Tokyo, Prof.N.Iwata of Tokyo Univ.of Merchantile, Prof.N.Shuto of Tohoku Univ., Prof.Y.Tsuji of the Univ.of Tokyo, Dr.Y.Fujinawa of National Research Center for Disaster Prevention, Dr.H.Nagashima of Physical and Chemical Research Institute, Dr.T.Hamada of Port and Harbour Research Institute and Dr.N.Moritani of Meteorological Bureau for giving me an amount of basic knowledge and new ideas through discussions and meetings of the《OCEAN WAVE RESEARCH GROUP》. In particular, Profs.Nagata and Tsuji have closely examined the manuscript of this paper and kindly given me many useful suggestions.

I thank many other persons, Dr.K.Sugai, the director general of S.R.I., for his continual encouragement, Dr.Y.Takaishi of the director of Ship Dynamics Division(S.D.D.), for giving me the convenience to make up this paper, Mr.H.Sawada and Mr.S.Ishida, colleages of S.D.D., for their assistance and consultation in the experiments, Mr.H.Kawaguchi, Miss.Y.Yokote, Mrs.Y.Uchino and Miss.S.Yamashita, for their help in preparing and refining many figures.

Appendix I Coefficients $H^{(N)}$, $F^{(N)}$

Coefficients in (1 − 24) are yielded as follows;

$$H^{(1)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = 1/(2\sqrt{2})\,[(gk_2/kk_1)^{1/4}D^{(1)}(\mathbf{k}_1, \mathbf{k}_2) + (gk/k_1k_2)^{1/4}D^{(2)}(\mathbf{k}_1, \mathbf{k}_2) - (gkk_1k_2)^{1/4}D^{(3)}(\mathbf{k}_1, \mathbf{k}_2)]\,\delta_{0-1-2}, \qquad (\mathrm{I}-1)$$

$$H^{(2)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = 1/(2\sqrt{2})\,[(gk_2/kk_1)^{1/4}D^{(1)}(\mathbf{k}_1, -\mathbf{k}_2) - (gk_1/kk_2)^{1/4}D^{(1)}(-\mathbf{k}^2, \mathbf{k}^1) - (gk/k_1k_2)^{1/4}\{D^{(2)}(-\mathbf{k}^2, \mathbf{k}^1) + D^{(2)}(\mathbf{k}_1, -\mathbf{k}_2)\} - (gkk_1k_2)^{1/4}\{D^{(3)}(\mathbf{k}_1, -\mathbf{k}_2) + D^{(3)}(-\mathbf{k}^2, \mathbf{k}^1)\}]\,\delta_{0+1-2}, \qquad (\mathrm{I}-2)$$

$$H^{(3)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = 1/(2\sqrt{2})\,[(gk_2/kk_1)^{1/4}D^{(1)}(\mathbf{k}_1, \mathbf{k}_2) + (gk/k_1k_2)^{1/4}D^{(2)}(\mathbf{k}_1, \mathbf{k}_2) - (gkk_1k_2)^{1/4}D^{(3)}(\mathbf{k}_1, \mathbf{k}_2)]\,\delta_{0+1+2}, \qquad (\mathrm{I}-3)$$

$$F^{(1)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = 1/4\,[(k_2k_3/kk_1)^{1/4}E^{(1)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) - (kk_3/k_1k_2)^{1/4}E^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + (kk_1k_2k_3)^{1/4}E^{(3)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)]\,\delta_{0-1-2-3}, \qquad (\mathrm{I}-4)$$

$$F^{(2)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = 1/4\,[(k_1k_2/kk_3)^{1/4}E^{(1)}(\mathbf{k}_3, \mathbf{k}_2, -\mathbf{k}_1) - (k_2k_3/kk_1)^{1/4}E^{(1)}(-\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + (k_1k_3/kk_2)^{1/4}E^{(1)}(\mathbf{k}_2, -\mathbf{k}_1, \mathbf{k}_3) +$$

$(kk_3/k_1k_2)^{1/4}\{E^{(2)}(\mathbf{k}_2, -\mathbf{k}_1, \mathbf{k}_3)+E^{(2)}(-\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\}-$

$(kk_1/k_2k_3)^{1/4}E^{(2)}(\mathbf{k}_3, \mathbf{k}_2, -\mathbf{k}_1)+$

$(kk_1k_2k_3)^{1/4}\{E^{(3)}(\mathbf{k}_3, \mathbf{k}_2, -\mathbf{k}_1)+E^{(3)}(\mathbf{k}_2, -\mathbf{k}_1, \mathbf{k}_3)+$

$E^{(3)}(-\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\}]\delta_{0+1-2-3}$.

(I−5)

$F^{(3)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)=1/4\,[(k_1k_2/kk_3)^{1/4}E^{(1)}(\mathbf{k}_3, -\mathbf{k}_2, -\mathbf{k}_1)$

$-(k_2k_3/kk_1)^{1/4}\{E^{(1)}(-\mathbf{k}_1, -\mathbf{k}_2, \mathbf{k}_3)+E^{(1)}(-\mathbf{k}_1, \mathbf{k}_3, -\mathbf{k}_2)\}$

$+(kk_2/k_1k_3)^{1/4}E^{(2)}(-\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_2)$

$+(kk_1/k_2k_3)^{1/4}E^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\}-$

$(kk_3/k_1k_2)^{1/4}E^{(2)}(-\mathbf{k}_1, -\mathbf{k}_2, \mathbf{k}_3)+$

$(kk_1k_2k_3)^{1/4}\{E^{(3)}(\mathbf{k}_3, \mathbf{k}_2, -\mathbf{k}_1)+E^{(3)}(\mathbf{k}_1, \mathbf{k}_3, -\mathbf{k}_2)+$

$E^{(3)}(-\mathbf{k}_1, -\mathbf{k}_2, \mathbf{k}_3)\}]\delta_{0+1+2-3}$

and (I−6)

$F^{(4)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)=$

$1/4\,[-(k_2k_3/kk_1)^{1/4}E^{(1)}(-\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3)+$

$(kk_3/k_1k_2)^{1/4}E^{(2)}(-\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3)+$

$(kk_1k_2k_3)^{1/4}E^{(3)}(-\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3)]\delta_{0+1+2+3}$,

(I−7)

where, $\delta_{0+1-2-3}=\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)$ and

$D^{(1)}(\mathbf{k}_1, \mathbf{k}_2)=\mathbf{k}_1\mathbf{k}_2+\mathbf{k}_1^2$,

$$D^{(2)}(\mathbf{k}_1,\ \mathbf{k}_2)=\tfrac{1}{2}(\mathbf{k}_1\mathbf{k}_2-k_1k_2),$$

$$D^{(3)}(\mathbf{k}_1,\ \mathbf{k}_2)=\tfrac{1}{2}(k_1+k_2),$$

$$E^{(1)}(\mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3)=\tfrac{1}{2}k_1\{k_1^2+\mathbf{k}_1(\mathbf{k}_2+\mathbf{k}_3)\},$$

$$E^{(2)}(\mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3)=-\tfrac{1}{2}(\mathbf{k}_1\mathbf{k}_2-k_1k_2)\ \{|\mathbf{k}_1+\mathbf{k}_2|$$
$$-(k_1^2+k_2^2)\},$$

$$E^{(3)}(\mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3)=-(1/6)\{(k_1+k_2)\ |\mathbf{k}_1+\mathbf{k}_2|$$
$$+(k_2+k_3)\ |\mathbf{k}_2+\mathbf{k}_3|$$
$$+(k_3+k_1)\ |\mathbf{k}_3+\mathbf{k}_1|$$
$$-(k_1^2+k_2^2+k_3^2)\}.$$

Appendix II On Canonical Form

As is well known, the energy of deep-water gravity waves is represented by

$$E=\frac{1}{2}\int_{S} d\mathbf{r}\left[\int_{-B}^{\eta}(\nabla\phi)^{2}dz+g\eta^{2}\right], \qquad (\text{II}-1)$$

where, S means the total surface considered here, B is the depth where the wave effect diminishes. This expression contains the volume integral over all region occupied with the fluid, it can be replaced by the surface integrals by means of certain transformation of variables as follows.

From the Gauss' theorem, the first term of the right hand side of (II − 1) (KINEMATIC ENERGY) is expressed by the next equation

$$E_{1}=\frac{1}{2}\int_{S}\phi\,(\partial\phi/\partial n)\,dS, \qquad (\text{II}-2)$$

when S means the fluid surface $z=\eta(\mathbf{r},\ t)$.
By use of the theorem of differential geometry, the relations

$$(\partial\phi/\partial n)_{\eta}=(\phi_{z}-\nabla_{h}\phi\nabla_{h}\eta)/\{1+(\nabla_{h}\eta)^{2}\}^{1/2}\Big|_{\eta}$$

and

$$dS=\{1+(\nabla_{h}\eta)^{2}\}^{1/2}\,d\mathbf{r}$$

are derived. Where, the operator ∇_h means the horizontal two-dimensional gradients.

Furthermore, if we consider the kinematic condition

$$\phi_{z}-\nabla_{h}\phi\nabla_{h}\eta=\eta_{t}$$

at $z=\eta$, (II − 2) is proved to be replaced by

$$E_{1}=\frac{1}{2}\int_{\Sigma}\phi^{s}\,(\partial\eta/\partial t)\,d\mathbf{r}. \qquad (\text{II}-3)$$

If we add the potential energy term to this, total energy H comes out to

$$H=\frac{1}{2}\int_{S}d\mathbf{r}\,[\phi^{s}\eta_{t}+g\eta^{2}], \qquad (\text{II}-4)$$

where, ϕ^{s} is the value of the potential at the fluid surface.

An alternative proof of this theorem is proposed by West(1981). The interpretation of H to be explained as the Hamiltonian of water waves accompanied with the canonical variables ϕ^s and η was presented by Miles(1977), Milder(1977).

To the Hamiltonian (II−4), the new variables p, q are defined by use of the Fourier transform as

$$\phi^s(\mathbf{r},\ t)=(2\pi)^{-1}\iint_{-\infty}^{\infty}d\mathbf{k}\,p\,(\mathbf{k},\ t)\ e^{i\mathbf{k}\mathbf{r}} \qquad (\text{II}-5)$$

and

$$\eta\,(\mathbf{r},\ t)=(2\pi)^{-1}\iint_{-\infty}^{\infty}d\mathbf{k}\,q\,(\mathbf{k},\ t)\ e^{i\mathbf{k}\mathbf{r}}. \qquad (\text{II}-6)$$

By use of them, H is represented by p, q as follows

$$H=\tfrac{1}{2}\int_{-\infty}^{\infty}d\mathbf{k}\ \{p^*(\mathbf{k})\ q_t(\mathbf{k})+g\,q^*(\mathbf{k})\ q\,(\mathbf{k})\}. \qquad (\text{II}-7)$$

In order to eliminate the function $q_t(\mathbf{k})$ from (II−7), we use the equations presented in Stiassnie & Shemer(1984) (they did not discuss the canonical form), that

$$q_t(\mathbf{k})=w^s+$$

$$\frac{1}{2\pi}\iint_{-\infty}^{\infty}(\mathbf{k}_1\cdot\mathbf{k}_2)\ d\mathbf{k}_1 d\mathbf{k}_2 p\,(\mathbf{k}_1)\ q\,(\mathbf{k}_2)\ \delta\,(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)$$

and

$$w^s=k\,p\,(\mathbf{k})-$$

$$\frac{1}{2\pi}\iint_{-\infty}^{\infty}k_1[k-k_1]\ d\mathbf{k}_1 d\mathbf{k}_2 p\,(\mathbf{k}_1)\ q\,(\mathbf{k}_2)\ \delta\,(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2),$$

where, we adopt the notations used here and truncated the perturbation series up to the term necessary in this discussion. The function w^s means the value of ϕ_z at the fluid surface.

We make q_t the function of p, q and substitute it into (II−7) and the representation

$$H=\tfrac{1}{2}\int_{-\infty}^{\infty}d\mathbf{k}\ \{k\,p^*(\mathbf{k})\ p\,(\mathbf{k})+g\,q^*(\mathbf{k})\ q\,(\mathbf{k})\}+$$

$$\frac{1}{4\pi}\iint_{-\infty}^{\infty} K^{(2)} dk\, dk_1 dk_2 p^*(k)\, p(k_1)\, q(k_2)\, \delta(k-k_1-k_2)$$

$$-\frac{1}{8\pi^2}\iiint_{-\infty}^{\infty} K^{(3)} dk\, dk_1 dk_2 dk_3 p^*(k)\, p(k_1)\, q(k_2)\, q(k_3)$$

$$\times\delta(k-k_1-k_2-k_3) \qquad (\mathrm{II}-8)$$

is derived. The first term is the well-known Hamiltonian of linear wave field. The kernels $K^{(2)}$ and $K^{(3)}$ are

$$K^{(2)}(k,\ k_1,\ k_2) = (k_1 k_2) - k_1(k-k_1), \qquad (\mathrm{II}-9)$$

$$K^{(3)}(k_1,\ k_2,\ k_3) = k_1(k_2 k_3). \qquad (\mathrm{II}-10)$$

Appendix III Randomization of the Zakharov equation

The arguments applied to the randomization of narrow-band wave system by Longuet-Higgins(1976) is extended to this problem of arbitrary band width wave system in the following.

The exact form of the Zakharov equation is presented by (3 − 1) in Chapter 3. Here, we abbrebiate it to the following form

$$i\frac{dB_0}{dt}=\int dK\,T\,B_1^{*}B_2B_3\delta\,e^{i\Delta t} \qquad (\text{III}-1)$$

Multiplying B_0^{*} to the both sides of the equation and subtracting it from its complex conjugate, we have

$$i\frac{d|B_0|^2}{dt}=2i\,\mathrm{Im}\int dK\,T\,B_1^{*}B_2B_3B_0^{*}\delta\,e^{i\Delta t} \qquad (\text{III}-2)$$

If we write the ensemble average of $|B_0|^2$ by $<|B_0|^2>=C_0$, we get from (III − 2) the statistical equation

$$i\frac{dC_0}{dt}=2i\,\mathrm{Im}\int dK\,\widetilde{T}\,C_0C_1\,. \qquad (\text{III}-3)$$

In this equation, the right-hand side is 0, because all the quantities in the integrant are real numbers. Therefore, the energy spectrum of the stochastic wave field does not vary to the 4-th order.

Finally, time derivative of the 4-th order mutual products are calculated as

$$i\,(B_0^{*}B_1^{*}B_2B_3)_t=i\,B_{0\,t}^{*}B_1^{*}B_2B_3+i\,B_0^{*}B_{1\,t}^{*}B_2B_3$$

$$+\,i\,B_0^{*}B_1^{*}B_{2t}B_3+i\,B_0^{*}B_1^{*}B_2B_{3t} \qquad (\text{III}-4)$$

Substituting (III − 1) to the time derivatives in the right hand side of (III − 4), averaging the whole equation, and remaining up to the 6-th order of magunitude B, it is yielded as

$$i<(B_0^{*}B_1^{*}B_2B_3)>_t=2T\,\{C_2C_1C_0+C_3C_1C_0$$

$$-C_0C_2C_3-C_1C_2C_3\}\,\delta\,\exp(-i\Delta t)\,. \qquad (\text{III}-5)$$

Rewriting (Ⅲ－2), we get the equation

$$i\frac{dC_0}{dt}=-2i\mathrm{Re}\int dK\,T\,i<B_1^*B_2B_3B_0^*>\delta\,e^{i\Delta t} \qquad (\text{Ⅲ}-6)$$

To evaluate the right hand side of this equation, (Ⅲ－5) is integrated to be

$$i<(B_0^*B_1^*B_2B_3)>=2\int_{-\infty}^{t}d\tau\,T\,\{\}\,\delta\,e^{-i\Delta\tau} \qquad (\text{Ⅲ}-7)$$

In this equation, {} denotes the quantity in the bracket in (Ⅲ－5). Therefore,

$$i\frac{dC_0}{dt}=-4i\mathrm{Re}\int dK\,T^2\,\{\}\,\delta\int_{-\infty}^{t}d\tau\;e^{i\Delta(t-\tau)} \qquad (\text{Ⅲ}-8)$$

is obtained.
The last definite integral is turn out to be $\pi\delta(\Delta)$, so that the final result has the form

$$\frac{dC_0}{dt}=4\pi\int dK\,T_{0123}^2\,\{C_2C_3(C_0+C_1)-C_0C_1(C_2+C_3)\}\,\delta_{0123}\delta(\Delta_{0123}), \qquad (\text{Ⅲ}-9)$$

which is just the same form with the energy transport equation among the spectral components first given by Hasselmann(1962, 1963a, 1963b).

AppendixⅣ. Narrow band approximation of the Zakharov equation

So called non-linear Schroedinger equation was first derived in the paper of Zakharov(1968) himself. In this Appendix, we interprete the procedure in terms of the symbols used in this paper.

We restrict ourselves that B has large value only in the vicinity of certain central wave-number $\mathbf{k}_0$. That is, $\mathbf{k}=\mathbf{k}_0+\Psi$, so

$$B(\mathbf{k})\exp\{i(\mathbf{k}\mathbf{r}-\omega t)\}=B(\mathbf{k})\exp\{i(\mathbf{k}_0\mathbf{r}+\Psi\mathbf{r}-\omega t+\omega_0 t+\omega_0 t)\}$$

$$\equiv A(\Psi)\exp(i\Psi\mathbf{r})\exp\{i(\mathbf{k}_0\mathbf{r}-\omega_0 t)\} \tag{Ⅳ-1}$$

is introduced to the fractional wave-number Ψ. Using this formula, the elevation η is expressed as

$$\eta=\left[\frac{1}{2\pi}(k_0/2\omega_0)^{\frac{1}{2}}\int_{-\infty}^{\infty}d\Psi A(\Psi)e^{i\Psi\mathbf{r}}\right]e^{i(\mathbf{k}_0\mathbf{r}-\omega_0 t)}+c.c. \tag{Ⅳ-2}$$

By definition, the quantity in 【】 is one half of the wave envelope a ($\mathbf{r}$, t), therefore, the relation of A to a is

$$a(\mathbf{r},t)=\frac{1}{2\pi}(k_0/2\omega_0)^{\frac{1}{2}}\int_{-\infty}^{\infty}d\Psi A(\Psi)e^{i\Psi\mathbf{r}}. \tag{Ⅳ-3}$$

On the other hand, (Ⅳ-1) is substituted to the Zakharov equation (Ⅲ-1) to be

$$i\frac{dA_0}{dt}-(\omega-\omega_0)A_0=\int dK\,T A_1^{*}A_2A_3\delta e^{i\Delta t}. \tag{Ⅳ-4}$$

Assuming that the band width $\Psi=(\phi,\lambda)$ is sufficiently small, we expand the dispersion relation $\omega=f(k)$ around ω_0 up to the 2-nd order of ϕ, λ to have

$$\omega-\omega_0=(\omega_0/2k_0)\,\phi-(\omega_0/8k_0^2)\,\phi^2+(\omega_0/4k_0^2)\,\lambda^2 . \qquad (\text{IV}-5)$$

Substituting this into (IV－4) and Fourier transforming with respect to $\mathbf{k}$, we get the final equation by use of the approximate kernel $T_{0123}=T_{0000}=k_0^3/4\pi^2$

$$i\left(\frac{\partial a}{\partial t}+c_g\frac{\partial a}{\partial x}\right)+\frac{\omega_0}{8k_0^2}\frac{\partial^2 a}{\partial x^2}-\frac{\omega_0}{4k_0^2}\frac{\partial^2 a}{\partial y^2}=\frac{\omega_0 k_0^2}{2}|a|^2 a . \qquad (\text{IV}-6)$$

This equation agrees with the 2-dimensional Nonlinear Schroedinger equation for deep-water gravity waves (see, Yuen & Lake(1982)) .

Modifications of (IV－6) for including the mean flow effects were proposed by Dysthe(1979) to deep-water waves and by Tomita(1985b, 1986) to waves on a finite water depth.

Appendix V Kernel T of the Zakharov equation

The third order interaction coefficient $T(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ appearing in (3 − 1) was first found by Zakharov(1968) and rederived by Crawford et.al.(1981) is exhibited below with some minor misprints removed:

$$T(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = T_{0123} =$$

$$-\frac{2V^{(-)}_{3,3-1,1}V^{(-)}_{0,2,0-2}}{\omega_{1-3}-\omega_3+\omega_1} - \frac{2V^{(-)}_{2,0,2-0}V^{(-)}_{1,1-3,3}}{\omega_{1-3}-\omega_1+\omega_3}$$

$$-\frac{2V^{(-)}_{2,2-1,1}V^{(-)}_{0,3,0-3}}{\omega_{1-2}-\omega_2+\omega_1} - \frac{2V^{(-)}_{3,0,3-0}V^{(-)}_{1,1-2,2}}{\omega_{1-2}-\omega_1+\omega_2}$$

$$-\frac{2V^{(-)}_{0+1,0,1}V^{(-)}_{2+3,2,3}}{\omega_{2+3}-\omega_2+\omega_3} - \frac{2V^{(+)}_{-2-3,2,3}\ V^{(+)}_{0,1,-0-1}}{\omega_{2+3}-\omega_2+\omega_3}$$

$$+W_{0,1,2,3}\ ,$$

where,

$$V^{(\pm)}_{0,1,2} = \frac{1}{8\pi\sqrt{2}}\left\{(\mathbf{k}_0\cdot\mathbf{k}_1 \pm k_0 k_1)\left[\frac{\omega_0\omega_1}{\omega_2}\ \frac{k_2}{k_0 k_1}\right]^{1/2}\right.$$

$$+(\mathbf{k}_0\cdot\mathbf{k}_2 \pm k_0 k_2)\left[\frac{\omega_0\omega_2}{\omega_1}\ \frac{k_1}{k_0 k_2}\right]^{1/2}$$

$$\left.+(\mathbf{k}_1\cdot\mathbf{k}_2 + k_1 k_2)\left[\frac{\omega_1\omega_2}{\omega_0}\ \frac{k_0}{k_1 k_2}\right]^{1/2}\right\},$$

$$\omega_{1+2} = \omega(\mathbf{k}_1 + \mathbf{k}_2)$$

and

$$W_{0,1,2,3} = \overline{W}_{-0,-1,2,3} + \overline{W}_{2,3,-0,-1} - \overline{W}_{2,-1,-0,3} - \overline{W}_{-0,2,-1,3} +$$

$$\overline{W}_{-0,302,-1} - \overline{W}_{3,-1,2,-0}$$

with

$$\bar{W}_{0,1,2,3} = \frac{1}{64\pi^2}\left[\frac{\omega_0\omega_1}{\omega_2\omega_3} k_0 k_1 k_2 k_3\right]^{1/2} \times$$
$$\{2(k_0 + k_1) - k_{1+3} - k_{1+2} - k_{0+3} - k_{0+2}\}$$

and $$k_{1+2} = |\mathbf{k}_1 + \mathbf{k}_2| \ .$$

Appendix VI Dispersion Relation in Tertiary Resonant Interaction

In general, the kernel $T\ (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ of the Zakharov equation is so complicated that the simple analytical expression was obtained only in the cases that $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3 = \mathbf{k}_4$ (single wave) and $\mathbf{k}_1 = -\mathbf{k}_2 = -\mathbf{k}_3 = \mathbf{k}_4$ (standing wave) in the paper by Okamura(1984). We deal with here the next simplest case that $\mathbf{k}_1 = \mathbf{k}_3 = k\ (\cos\theta,\ \sin\theta)$ and $\mathbf{k}_2 = \mathbf{k}_4 = k\ (\cos\theta,\ -\sin\theta)$.

If there exists only two trains of wave of exactly same amplitude a and wavelength $\lambda = 2\pi / k$ intersecting by the angle 2θ, the equations corresponding to (3 − 3) become

$$i\frac{dB_1}{dt} = \{T_{1111}B_1{}^*B_1 + \widetilde{T}_{1221}B_2{}^*B_2\}\ B_1 \qquad (\text{VI}-1-1)$$

and

$$i\frac{dB_2}{dt} = \{\widetilde{T}_{2112}B_1{}^*B_1 + T_{2222}B_2{}^*B_2\}\ B_2\,. \qquad (\text{VI}-1-2)$$

These are easily solved by setting $B_1 = b\exp(-i\chi_1)$, $B_2 = b\exp(-i\chi_2)$ with real constant b. From (VI − 1) and (3 − 5), $\chi_{1,2}$ are given by

$$\chi_1 = \{T_{1111} + \widetilde{T}_{1221}\}\ (2\pi^2\omega/k)\ a^2 \qquad (\text{VI}-2-1)$$

and

$$\chi_2 = \{\widetilde{T}_{2112} + T_{2222}\}\ (2\pi^2\omega/k)\ a^2\,. \qquad (\text{VI}-2-2)$$

The resultant of the two waves are called the SHORT CRESTED WAVE of amplitude $A = 2a$ and its dispersion relation was derived by Mollo-Christensen(1981) that

$$\omega = \omega_0\ \{1 + \tfrac{1}{4}A^2k^2F\ (\theta)\} \qquad (\text{VI}-3)$$

where

$$F\ (\theta) = (8\cos^2\theta - 3 - 2\cos^4\theta)/2 + \sin^2\theta\,(\cos\theta + 2 - 4\cos^2\theta)/(2-\cos\theta). \qquad (\text{VI}-4)$$

These formula are also verified from the equation (2 − 8) of Longuet-Higgins & Phillips(1962), after some minor correction.

We can rederive this result by means of Zakharov theory that

$$T_{1111}+\widetilde{T}_{1221}=\widetilde{T}_{2112}+T_{2222}=(k^3/2\pi^2)\,F(\theta) \qquad (\text{VI}-5)$$

after some algebraic manipulation. At least in these simple cases, it is revealed that the Zakharov theory yields an identical result with the classical one (see Tomita(1985a)) .

Appendix VII Conservation laws of the Zakharov equation

Here, we define two quantities E and C such that

$$E_i = g A_i^2 / 2 = \omega_i B_i B_i^* / (2\pi)^2 \qquad (\text{VII}-1)$$

$$C_i = E_i / \omega_i = B_i B_i^* / (2\pi)^2 \qquad (\text{VII}-2)$$

They are called the energy and the wave action of waves. Considering the total energy of three waves that

$$E = E_1 + E_2 + E_3 = \{\omega_1 B_1 B_1^* + \omega_2 B_2 B_2^* + \omega_2 B_2 B_2^*\} / (2\pi)^2 ,$$

we obtain the following expression to its derivative dE/dt by use of (3−3−1) ∼ (3−3−3),

$$i \frac{dE}{dt} = \{\omega_1 (\tilde{T}_{1123} e^{i\Delta t} B_1^{*2} B_2 B_3 - c.c.) +$$

$$\omega_2 (T_{2311} e^{-i\Delta t} B_1^2 B_2^* B_3^* - c.c.) +$$

$$\omega_3 (T_{2311} e^{-i\Delta t} B_1^2 B_2^* B_3^* - c.c.) \} / (2\pi)^2$$

$$= [\{\omega_1 \tilde{T}_{1123} - \omega_2 T_{2311} - \omega_3 T_{3211}\} e^{i\Delta t} B_1^{*2} B_2 B_3 - c.c.] / (2\pi)^2,$$

where c.c. signifies the complex conjugate of preceding term.

Because the equalities $\tilde{T}_{1123} = 2T_{2311} = 2T_{3211} = 2T$ holds with the resonance condition $2\omega_1 - \omega_2 - \omega_3 = 0$, the change of energy is

$$\frac{dE}{dt} = \mathrm{Im}[\{2\omega_1 - \omega_2 - \omega_3\} T e^{i\Delta t} B_1^{*2} B_2 B_3] / (2\pi)^2 = 0 . \qquad (\text{VII}-3)$$

Thus, the total energy conservation is proved in the present situation. The conservation of total wave action C is also derived by the method akin to the above procedure. It is in contrast to the case of capillary-gravity waves (for reference Leibovich & Seebas(1974) or Whitham(1974)) .The conservation of wave action leads to next simultaneous equations with respect to the amplitudes A_i,

$$g A_1^2/\omega_1 + g A_2^2/\omega_2 + g A_3^2/\omega_3 = \text{const}$$

and

$$g A_2^2/\omega_2 - g A_3^2/\omega_3 = \text{const} \quad .$$

Initial values of $A_1 = A_{10}$, $A_2 = A_{20}$, $A_3 = 0$ are substituted to the right-hand constant terms and the elimination of A_2 leads to

$$A_1^2/\omega_1 + 2 A_3^2/\omega_3 = A_{10}^2/\omega_1 \quad .$$

It means that the capable maximum amplitude of tertiary wave has a limit

$$A_3 \leqq (\omega_3/2\omega_1)^{1/2} A_{10} = (2\gamma - 1/2\gamma)^{1/2} A_{10} = 0.844 A_{10} \quad . \tag{VII-4}$$

In order to make the condition of validity of the Zakharov equation clear, we estimate (1-10) with respect to γ. The condition

$$\Delta = 2\omega_1 - \omega_2 - \omega_3 \sim 0 \tag{VII-5}$$

is to be evaluated. This is rewritten as

$$\Delta = \omega_0 - \omega_3 \sim \varepsilon^2 \omega_3 \tag{VII-6}$$

using $\omega_0 = 2\omega_1 - \omega_2$. Small quantity Δ is estimated that

$$\Delta = (g k_0)^{1/2} - (g k_3)^{1/2}$$

$$= -\frac{1}{2} (g/k_3)^{1/2} (k_0 - k_3) = \frac{1}{2} \omega_3 2 \delta k .$$

By virtue of (1-11), we see that $2\delta k = -\beta k_3 \delta\gamma$ where $\delta\gamma = \gamma - \gamma_0$. Substituting it to the relation,

$$\Delta = -\frac{1}{2} \omega_3 \beta \delta \gamma \tag{VII-7}$$

is yielded. Thus, from (VI-6), $\frac{1}{2}\beta \mid \delta\gamma \mid \sim \varepsilon^2$ results. Using the constant value $\beta = 0.497 \sim 0.5$ from the theory and adopting the small quantity $\varepsilon \sim A k$ to be 0.2 from the experiment, we obtain an estimation of γ such that

$$|\delta\gamma| \sim 0.16, \quad \text{i.e.,} \quad 1.58 < \gamma < 1.90 \qquad (\text{VII}-8)$$

Although the estimation examined above is not always accurate, the range of γ is verified almost to cover that of experiment in this paper.

Appendix VIII On the similarity of the theories

The Zakharov equation in Chapter 3 has the dimensional form and the calculations are conducted to the wavelengths comparable with the magnitude used in the experiment. However, all the results obtained in this paper must be applicable to the scale of actual ocean. In order to show this, we derive the non-dimensional form of the Zakharov equation and the classical solution.

First, we see from (3 − 2) and (3 − 3) the dimensions of B and T to be $B = [m^{3/2} s^{-1/2}]$, $T = [m^{-3}]$. Thus, we introduce the non-dimensional variables such as:

$$\omega_n = \omega_R w_n , \qquad (\text{VIII} - 1 - 1)$$

$$\omega_R t = \tau , \qquad (k_R = \omega_R^2 / g) , \qquad (\text{VIII} - 1 - 2)$$

$$T_{1234} = T_R U_{1234} , \qquad (\text{VIII} - 1 - 3)$$

$$(T_R = T(\mathbf{k}_R, \mathbf{k}_R, \mathbf{k}_R, \mathbf{k}_R) = k_R^3 / 4\pi^2) ,$$

$$B_n = B_R F_n , \qquad (\text{VIII} - 1 - 4)$$

$$(B_R = (2\omega_R / k_R)^{1/2} A_R) .$$

Substituting them in (3 − 3 − 1) for example, we have a non-dimensional form of the equation

$$i \frac{dF_1}{dt} = \mu \, [\{U_{1111} F_1^* F_1 + U_{1221} F_2^* F_2 + U_{1331} F_3^* F_3\} F_1 + \widetilde{U}_{1123} e^{i\delta\tau} F_1^* F_2 F_3] \qquad (\text{VIII} - 2)$$

where, $\delta = w_1 + w_2 + w_3 + w_4$ and $\mu = \omega_R^{-1} B_R^2 T_R$ is a non-dimensional constant. Connecting the relations (VIII − 1) together, coefficient μ is estimated as

$$\mu = \frac{1}{2} (A_R k_R)^2 . \qquad (\text{VIII} - 3)$$

This is nothing but a wave steepness.

By the similar manner, the classical solution (2 − 3) is written to the non-dimensional form

$$A_3K_3 = \frac{1}{2}(A_1K_1)^2(A_2K_2)(\omega_1 t)F(\gamma)(2-\gamma)^2 . \tag{VIII-4}$$

In case of perpendicular waves, non-dimensional coefficient is calculated to be $F(\gamma)(2-\gamma)^2 = 0.633$.

AppendixIX Analysis of Interaction Equations

1 Construction of Single Equation

We wright down again the interaction equations (3 − 3 − 1) ∼ (3 − 3 − 3) such that

$$i\frac{dB_1}{dt}=[T_{11}b_1^2+T_{12}b_2^2+T_{13}b_3^2]\ B_1+T_1B_1^*B_2B_3e^{i\Delta t},\qquad (\mathrm{IX}-1-1)$$

$$i\frac{dB_2}{dt}=[T_{21}b_1^2+T_{22}b_2^2+T_{23}b_3^2]\ B_2+T_2B_3^*B_1B_1e^{-i\Delta t}\qquad (\mathrm{IX}-1-2)$$

and

$$i\frac{dB_3}{dt}=[T_{31}b_1^2+T_{32}b_2^2+T_{33}b_3^2]\ B_3+T_3B_2^*B_1B_1e^{-i\Delta t}\qquad (\mathrm{IX}-1-3)$$

in which $b_n^2=B_nB_n^*$ and $\Delta=\omega_1+\omega_1-\omega_2-\omega_3$. Interaction coefficients T_n and symmetric matrix elements $[T_{kl}]=[T_{lk}]$ are real constants to be calculated from wave-numbers. The method of solution adopted here is that used by McGoldrick(1972) for second order non-linear equations in the context of capirally-gravity waves.

Multiplying B_1^* to (IX − 1 − 1) we obtain

$$iB_1^*\frac{dB_1}{dt}=[T_{11}b_1^2+T_{12}b_2^2+T_{13}b_3^2]\ b_1^2+T_1B_1^*B_1^*B_2B_3e^{i\Delta t}.$$

Taking the complex conjugate of this equation such as

$$-iB_1\frac{dB_1^*}{dt}=[T_{11}b_1^2+T_{12}b_2^2+T_{13}b_3^2]\ b_1^2+T_1B_1B_1B_2^*B_3^*e^{-i\Delta t},$$

and subtracting this from the former equation, it reduces to

$$i\frac{db_1^2}{dt}=T_1(R-R^*).\qquad (\mathrm{IX}-2-1)$$

In this expression, a complex quantity R is introduced such as

$$R = B_1^* B_1^* B_2 B_3 \exp(i \Delta t) \quad .$$

Similar relations are obtained by using (IX−1−2), (IX−1−3) that

$$i \frac{d b_2^2}{d t} = -T_2 (R - R^*) \qquad \text{(IX}-2-2\text{)}$$

$$i \frac{d b_3^2}{d t} = -T_3 (R - R^*) \qquad \text{(IX}-2-3\text{)}$$

From the relations (IX−2−1) ∼ (IX−2−3) we have three integrals

$$b_1^2/T_1 + b_2^2/T_2 = \mathrm{const}_1 = \acute{b}_1^2/T_1 + \acute{b}_2^2/T_2 \qquad \text{(IX}-3-1\text{)}$$

$$b_1^2/T_1 + b_3^2/T_3 = \mathrm{const}_2 = \acute{b}_1^2/T_1 + \acute{b}_3^2/T_3 \qquad \text{(IX}-3-2\text{)}$$

$$b_2^2/T_2 - b_3^2/T_3 = \mathrm{const}_3 = \acute{b}_2^2/T_2 - \acute{b}_3^2/T_3 \qquad \text{(IX}-3-3\text{)}$$

where $\acute{b}_n = b_n(0)$, (n=1, 2, 3), the initial value of $b_n(t)$.
By use of these integral properties, a complex function $Z(t)$ is introduced such as

$$Z(t) \equiv (\acute{b}_1^2 - b_1^2)/T_1 = (b_2^2 - \acute{b}_2^2)/T_2 = (b_3^2 - \acute{b}_3^2)/T_3.$$

$$\text{(IX}-4\text{)}$$

We can easily calculate that

$$d Z/d t = i (R - R^*) = -2 \mathrm{Im}(R) \qquad \text{(IX}-5\text{)}$$

In order to calculate the real part of R, we differentiate R with respect to t, that is,

$$d R/d t = 2 B_1^*{}_t B_1^* B_2 B_3 \exp(i \Delta t) + B_1^{*2} B_{2t} B_3 \exp(i \Delta t)$$

$$+ B_1^{*2} B_2 B_{3t} \exp(i \Delta t) + i \Delta B_1^{*2} B_2 B_3 \exp(i \Delta t) \ .$$

Substituting (IX−1−1) ∼ (IX−1−3) to this expression, it is yielded that

$$dR/dt = i\Delta R + 2i\,[T_{11}b_1^2 + T_{12}b_2^2 + T_{13}b_3^2]\,R + 2iT_1 b_1^2 b_2^2 b_3^2$$

$$-i\,[T_{21}b_1^2 + T_{22}b_2^2 + T_{23}b_3^2]\,R - iT_2 b_1^4 b_3^2$$

$$-i\,[T_{31}b_1^2 + T_{32}b_2^2 + T_{33}b_3^2]\,R - iT_3 b_1^4 b_2^2 \quad .$$

Taking the complex conjugate of this equation and adding them together, the result is expressed by

$$d\,(R + R^*)\,/dt = i\Delta\,(R - R^*)$$

$$+2i\,[T_{11}b_1^2 + T_{12}b_2^2 + T_{13}b_3^2]\,(R - R^*)$$

$$-i\,[T_{21}b_1^2 + T_{22}b_2^2 + T_{23}b_3^2]\,(R - R^*)$$

$$-i\,[T_{31}b_1^2 + T_{32}b_2^2 + T_{33}b_3^2]\,(R - R^*)\ .$$

Considering the relation (IX−5), it is transformed to

$$d\,(R + R^*)\,/dt = \{\Delta + \hat{T}_1 b_1^2 + \hat{T}_2 b_2^2 + \hat{T}_3 b_3^2\}\,dZ/dt$$

where $\hat{T}_n = 2T_{1n} - T_{2n} - T_{3n}$. Next, b_n^2 $(n=1,2,3)$ is eliminated by use of (IX−4), and we have

$$d\,(R + R^*)\,/dt = \{\Delta + \hat{T}_1\,(\acute{b}_1^2 - T_1 Z) + \hat{T}_2\,(\acute{b}_2^2 + T_2 Z)$$

$$+\hat{T}_3\,(\acute{b}_3^2 + T_3 Z)\}\,dZ/dt\ .$$

In this formula, direct integration is possible such that

$$2\mathrm{Re}\,(R) = R + R^* = H + \{\Delta + \hat{T}_1\acute{b}_1^2 + \hat{T}_2\acute{b}_2^2 + \hat{T}_3\acute{b}_3^2\}\,Z$$

$$-\frac{1}{2}\{\hat{T}_1 T_1 - \hat{T}_2 T_2 - \hat{T}_3 T_3\}\,Z^2 \qquad \text{(IX}-7\text{)}$$

where H is a real constant determined by initial conditions.
In order to fulfil the apparent equality that

$$|R|^2 = \{\mathrm{Re}\,(R)\}^2 + \{\mathrm{Im}\,(R)\}^2\ ,$$

The relations (IX−5) and (IX−7) are connected to

$$4(\acute{b}_1{}^2-T_1Z)^2(\acute{b}_2{}^2+T_2Z)(\acute{b}_3{}^2+T_3Z)$$

$$=(H+\xi Z+\eta Z^2)^2+(dZ/dt)^2 \qquad (\text{IX}-8)$$

where ξ and η are the coefficients determined in (IX−7).

2 Analysis of Resonant Growth

In the case that tertiary wave component does not exist initially, we can set the constant $H=0$ and $\acute{b}_3{}^2=0$ in (IX−8) so that we investigate the equation of the form

$$(dZ/dt)^2=f(Z) \qquad (\text{IX}-9)$$

where f is a quaritic function of Z such as

$$f(Z)=4(\acute{b}_1{}^2-T_1Z)^2(\acute{b}_2{}^2+T_2Z)T_3Z$$

$$-[\{\Delta+\hat{T}_1\acute{b}_1{}^2+\hat{T}_2\acute{b}_2{}^2\}-\tfrac{1}{2}\{\hat{T}_1T_1-\hat{T}_2T_2-\hat{T}_3T_3\}Z]^2Z^2.$$

$$(\text{IX}-10)$$

In general, real solution Z exists and can be solved by means of a integration

$$\int_0^t dt=\int_0^Z \frac{dx}{\sqrt{f(x)}} \qquad (\text{IX}-11)$$

if $f(x)$ is positive at $0<x\leqq Z$.

In order to obtain a formal solution, we must rearrange the polynomial $f(x)$ in its standard form such as (see Jeffreys & Jeffreys (1972)),

$$f(x)=a_0x^4+a_1x^3+a_2x^2+a_3x=\phi(x)$$

and it is resolved to the factors such that

$$\phi(x)=\phi_1(x)\phi_2(x)$$

where

$$\phi_1(x)=ax^2+bx+c \quad \text{and} \quad \phi_2(x)=x^2+\beta x .$$

A bilinear transformation of the variable is performed by

$$x=(Ay+B)/y+1 , \qquad (\mathrm{IX}-12)$$

in which A and B are real roots of the following equation,

$$(b-a\beta)\zeta^2+2c\zeta+c\beta=0 . \qquad (\mathrm{IX}-13)$$

In this procedure, integrant of (IX−11) is transformed as

$$\frac{dx}{\sqrt{\phi(x)}}=\frac{(A-B)\,dy}{\sqrt{P\{y^2+M\}\{y^2+N\}}} . \qquad (\mathrm{IX}-14)$$

There are several cases according to the signs of $P=\phi(A)$, $M=\phi_1(B)/\phi_1(A)$ and $N=\phi_2(B)/\phi_2(A)$.

Case I ; $P>0$, $M=\mu^2>0$, $N=-\nu^2<0$
In this case, (IX−14) is rewritten by

$$F(y)\,dy=\frac{(A-B)\,dy}{\sqrt{P\{y^2+\mu^2\}\{y^2-\nu^2\}}} . \qquad (\mathrm{IX}-15)$$

Transformation $y^2=\nu^2/(1-u^2)$ is adopted and

$$F(y)\,dy=\frac{(A-B)\,du}{\sqrt{P\{\mu^2+\nu^2\}(1-u^2)(1-k^2u^2)}}$$

$$(\mathrm{IX}-16)$$

results in the form of elliptic integral of first kind after some manipulation. In this formula, $k^2=\mu^2/\{\mu^2+\nu^2\}$ is called the generatrix of the integral.
Defining $\Omega=\sqrt{P\{\mu^2+\nu^2\}}/(A-B)$, integral (IX−11) reduces to

$$\Omega t=\int_{u_0}^{u}\frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}} \qquad (\mathrm{IX}-17)$$

and

$$u^2=1-\nu^2(A-Z)^2/(B-Z)^2 . \qquad (\mathrm{IX}-18)$$

From (IX－17), we obtain

$$u = \mathrm{sn}\,(\Omega t + \theta\ ;\ k^2)$$

and from (IX－18),

$$(A - Z)\,/\,(B - Z) = \nu^{-1}\,\mathrm{cn}\,(\Omega t + \theta\ ;\ k^2) \qquad (\text{IX}-19)$$

in which sn and cn are the Jacobi's elliptic functions.
Thus, the formal solution of (IX－9) is expressed by

$$Z = \frac{A - \mathrm{sig}\,(B)\ B\nu^{-1}\,\mathrm{cn}\,(\Omega t + \theta\ ;\ k^2)}{1 - \mathrm{sig}\,(B)\ \nu^{-1}\,\mathrm{cn}\,(\Omega t + \theta\ ;\ k^2)}\ , \qquad (\text{IX}-20)$$

where sig (B) means the signum of B.
To satisfy the initial condition that $Z = 0$ at $t = 0$, constant θ is determined by

$$A - \mathrm{sig}\,(B)\ B\nu^{-1}\,\mathrm{cn}\,(\theta\ ;\ k^2) = 0\ . \qquad (\text{IX}-21)$$

An example of this solution is shown in Fig－A－1. In this Figure, the variation of resonant wave amplitude A_3 is described under the conditions that $A_1 = 4$cm and $A_2 = 5$cm initially with $\gamma = 1.80$. The solid line is the solution obtained by the method discussed here. The symbol ○ is the numerical solution obtained in Chapter 3 (Fig－3－3 (c)). Both results which are obtained independently, coincide appreciably. Precision of the numerical procedure adopted in Chapter 3 is confirmed to be sufficient.

Case II ; $P > 0$, $M = -\mu^2 < 0$, $N = -\nu^2 < 0$

In this case, (IX－14) is rewritten by

$$G\,(y)\,dy = \frac{(A - B)\,dy}{\sqrt{P\,\{y^2 - \mu^2\}\,\{y^2 - \nu^2\}}}\ . \qquad (\text{IX}-22)$$

Transformation $y^2 = \nu^2 / u^2$ is adopted this time and

$$G\,(y)\,dy = \frac{(B - A)\,du}{\sqrt{P\nu^2\,(1 - u^2)\,(1 - k^2u^2)}}$$

(IX－23)

results also in the form of elliptic integral and $k^2=\mu^2/\nu^2$.
By the same procedure as in Case I, with $\Omega=\sqrt{P\nu^2}/(B-A)$ we have

$$Z=\frac{A+B\nu^{-1}\mathrm{sn}(\Omega t+\theta;k^2)}{1+\nu^{-1}\mathrm{sn}(\Omega t+\theta;k^2)} \quad . \qquad (\mathrm{IX}-24)$$

To satisfy the initial condition that $Z=0$ at $t=0$, constant θ is determined by

$$A+B\nu^{-1}\mathrm{sn}(\theta;k^2)=0 \quad . \qquad (\mathrm{IX}-25)$$

The transition from Case I to Case II occures under the condition of maximum growth of tertiary resonant wave which is clearly shown also by the numerical solution discussed in Ch3 of this paper.

3 Non-Periodic Solution

If we change the initial condition $\acute{b}_1^2$ or $\acute{b}_2^2$, two types of solution appear as interpreted in the previous section. Although both types of solution are periodic, there exist an aperiodic solution just at the critical region between Case I and Case II.

Returning to (IX−10), if the relation

$$\{\Delta+\hat{T}_1\acute{b}_1^2+\hat{T}_2\acute{b}_2^2\}-\tfrac{1}{2}\{\hat{T}_1T_1-\hat{T}_2T_2-\hat{T}_3T_3\}\acute{b}_1^2/T_1=0$$

$$(\mathrm{IX}-26)$$

is assumed to be realized, that is, the parameter $\acute{b}_1^2$, say, is sought so as to satisfy the following equation to the fixed $\acute{b}_2^2$, T_n, $\hat{T}_n$ (n=1, 2, 3) and Δ

$$\Delta=-\hat{T}_2\acute{b}_2^2-\tfrac{1}{2}\{\hat{T}_1T_1+\hat{T}_2T_2+\hat{T}_3T_3\}\acute{b}_1^2/T_1 \ ,$$

$$(\mathrm{IX}-27)$$

the equation $f(Z)=0$ has a double root at $Z=\acute{b}_1^2/T_1=\beta$ and $f(Z)$ is represented by

$$f(Z)=-aZ(Z-\beta)^2(Z-\gamma) \qquad (\mathrm{IX}-28)$$

where

$$a = -4T_1^2 T_2 T_3 + \frac{1}{4}\{\hat{T}_1 T_1 + \hat{T}_2 T_2 + \hat{T}_3 T_3\}^2 > 0,$$

$$\beta = \hat{b}_1^2 / T_1 > 0$$

and

$$\gamma = 4T_1^2 T_3 \hat{b}_2^2 / a > 0$$

are the positive constants in this situation with $\beta < \gamma$.

In this special case, (IX−9) is easily solved and the non-periodic solution is obtained as follows,

$$Z = \frac{\beta\gamma \tanh^2 \lambda t}{(\gamma - \beta) + \beta \tanh^2 \lambda t} \qquad \text{(IX−29)}$$

where $\lambda = \{a\beta(\gamma - \beta)\}^{1/2}/2$.

It is remarkable that Z approaches a constant β when t goes to infinity and all the energy initially contained in the first primary wave is transferred monotonically to the other components. Note that maximum amplitude realized by tertiary resonant wave a_3 is determined only by the initial value of the first primary wave amplitude a_1 and is independent of a_2 as discussed in Ch3. The condition (IX−27) is fulfiled even $\Delta = 0$ (exact resonance $\gamma = 1.736$). In this condition, the ratio of amplitudes of two primary waves is determined $a_2/a_1 = 3.1655\cdots\cdots$. To the values computed numerically in Ch3, it corresponds that $a_1 = 1.5795\cdots\cdots$cm and the asymptotic growth of tertiary wave would be $a_3 = 1.332\cdots\cdots$cm which are consistent with the numerical results.

For the case of wave instability problem, we can apply this theory by the following manner. This time b_1^2 is a primary wave and $b_2^2 = b_3^2 = b_s^2$ are two side band components recognized as small perturbations. To the leading order, $T_1 = T_2 = T_3 = T = k_1^3/4\pi^2$, $\hat{T}_1 = \hat{T}_2 = \hat{T}_3 = 0$ and $\Delta = 0$ so that (IX−9) and (IX−10) are reduced to

$$(dZ/dt)^2 = 4T^4 (Z - \beta)^2 (Z + \gamma)^2 . \qquad \text{(IX−30)}$$

where $\beta = \hat{b}_1^2/T$, $\gamma = \hat{b}_s^2/T$ and $\beta \gg \gamma$. This equation is easily solved as

$$Z = \gamma\{\exp(2\beta T^2 t) - 1\}$$

and evolution of the amplitude of side band components is expressed in terms of the steepness of primary wave such that

$$a_s(t) = a_{s0} \exp\{\tfrac{1}{2}(a_1 k_1)^2 \omega_1 t\} . \qquad (\text{IX}-31)$$

The growth rate of side band components $\frac{1}{2}(a_1 k_1)^2 \omega_1$ obtained in this theory is in accordance with the Benjamin-Feir(1967) theory.

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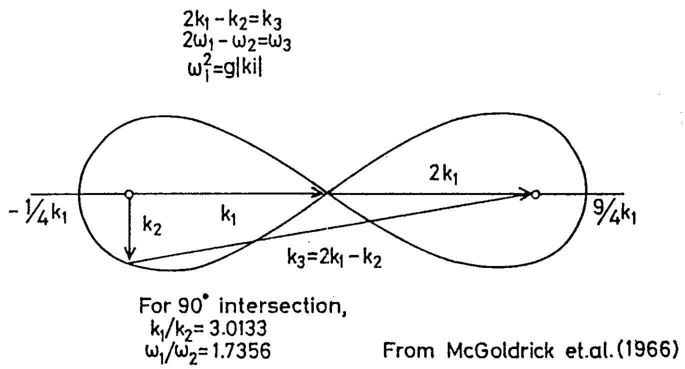


Fig. -1-1
Resonance curve;
Solutions to the resonance conditions.
K_1 : first-primary wave
K_2 : second-primary wave
K_3 : tertiary resonant wave

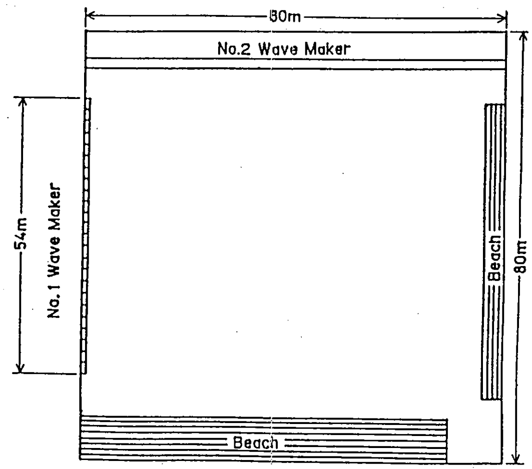


Fig. -2-1
Plan of the basin.
80m (length) × 80m (width) × 4.5m (depth)

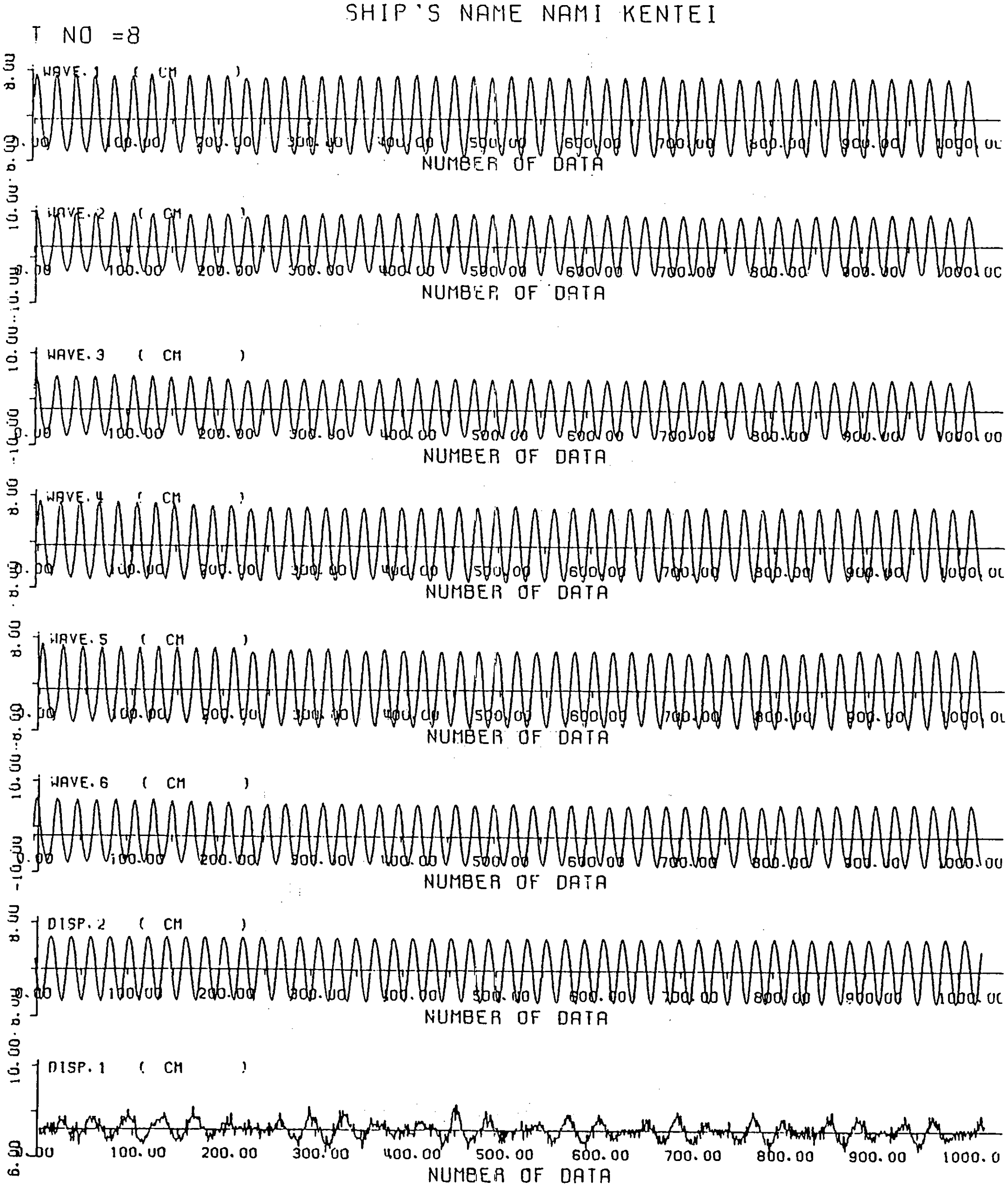


Fig. -2-2 (a)
Examples of the mesurement.
first primary wave is generated.
Upper six rows are wave records.
Lower two rows are records of stroke
of wave-makers.

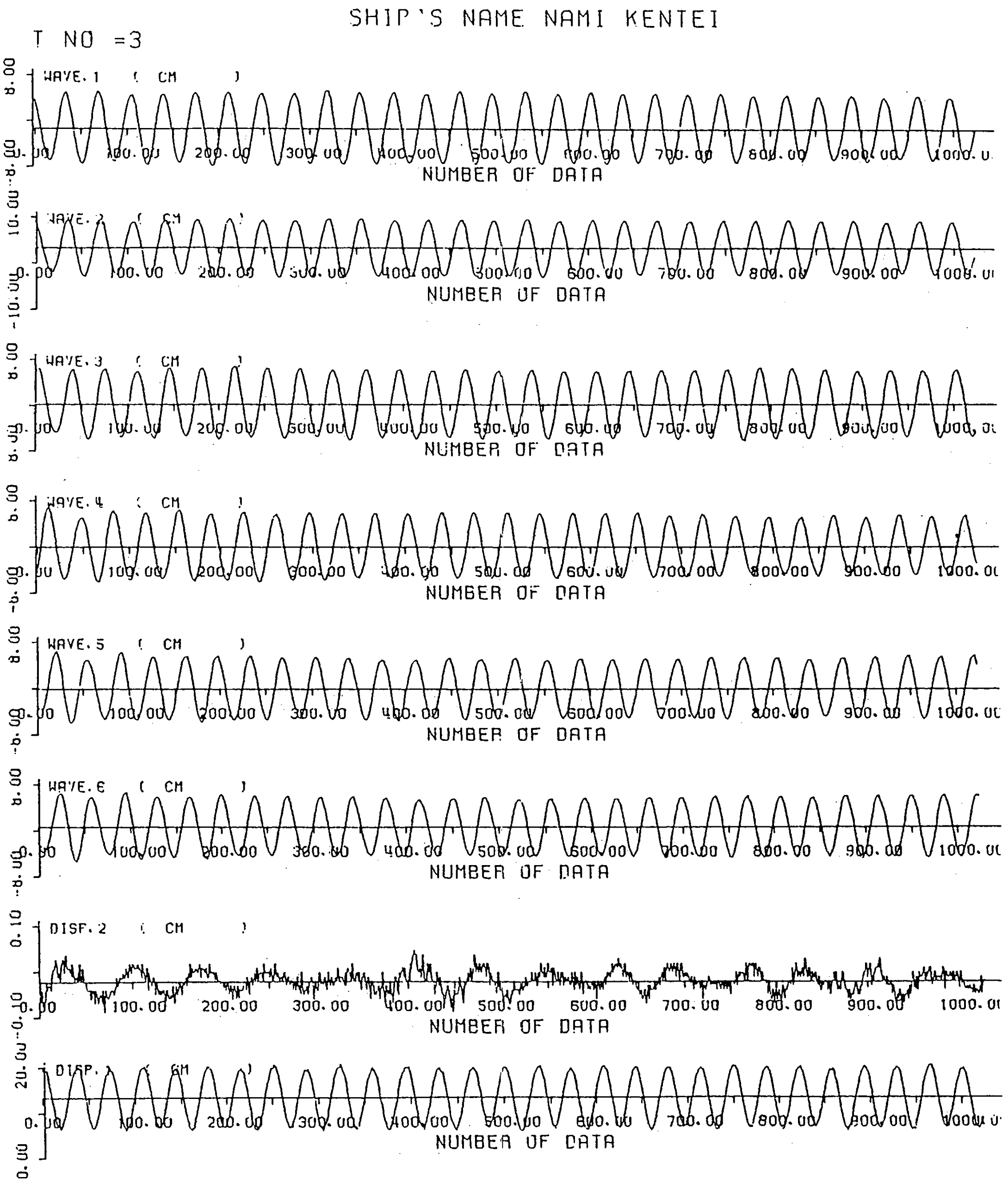


Fig. -2-2 (b)
Examples of the mesurement.
second primary wave is generated.
Upper six rows are wave record.
Lower two rows are records of stroke
of wave-makers.

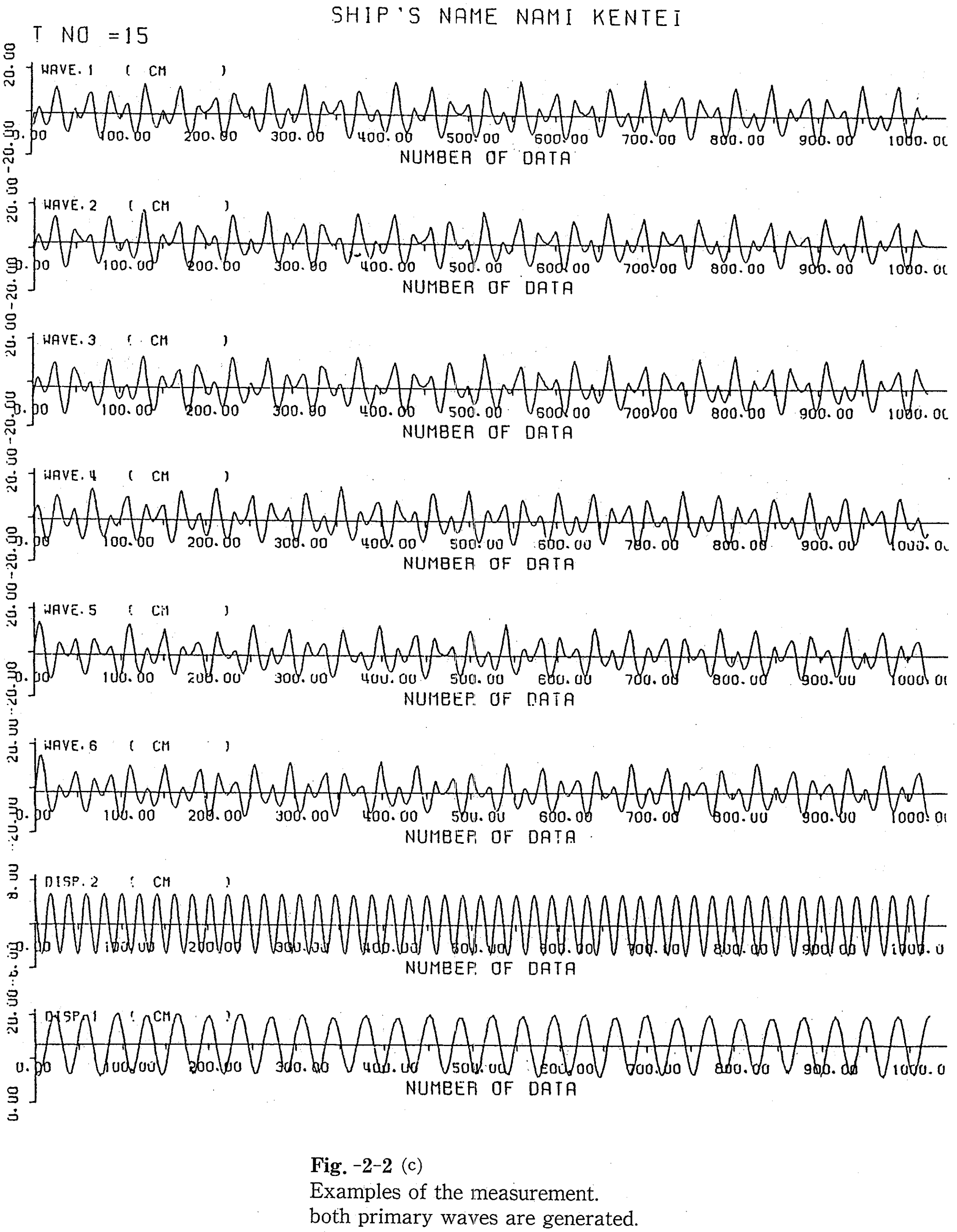


Fig. -2-2 (c)
Examples of the measurement.
both primary waves are generated.
Upper six rows are wave records.
Lower two rows are records of stroke
of wave-makers.

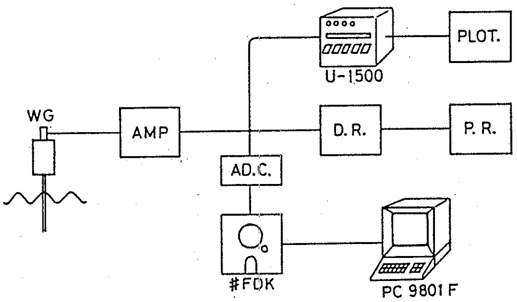


Fig. -2-3
Data collection system.
WG (wave gauge), AMP (amplifier), AD. C (AD converter)
D. R. (data recorder), P.R. (printer), PLOT(plotter)
FDK (disquet)

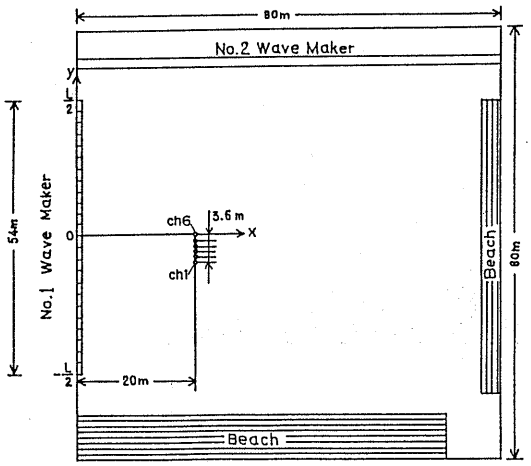


Fig. -2-4
Arrangement of the wave gauges (Case I)
For analysing the short term growth and the direction of the resonant waves.

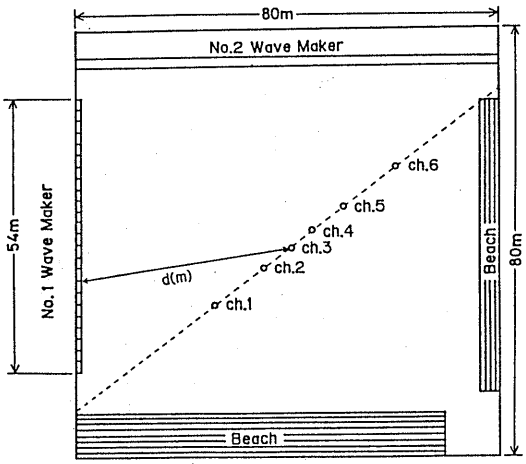


Fig. -2-5
Arrangement of the wave gauges (Case II).
For analysing the long term growth of the resonant waves.

Table-2-1 Elements of Mechanically Generated Waves

1-ST PRIMARY WAVE		2-ND PRIMARY WAVE		
PERIOD	WAVE HEIGHT	PERIOD	WAVE HEIGHT	γ
0.93	3～13	1.77	2.5～10	1.897
0.96				1.845
0.99				1.793
1.02				1.724
1.10	3～13	2.09	～5	1.898
1.15				1.816
1.19				1.755

PERIOD(sec), WAVE HEIGHT(cm), $\gamma = \omega_1 / \omega_2$

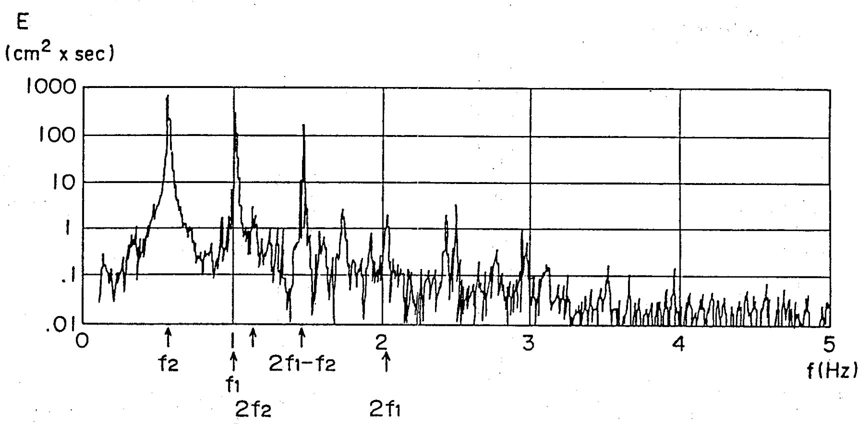


Fig. -2-6
An example of power spectrum. γ=1.793, d=45m
f_1 : 1-st primary wave, $2f_1$: 2-nd harmonics
f_2 : 2-nd primary wave, $2f_2$: 2-nd harmonics
$2f_1$-f_2 : tertiary resonant wave

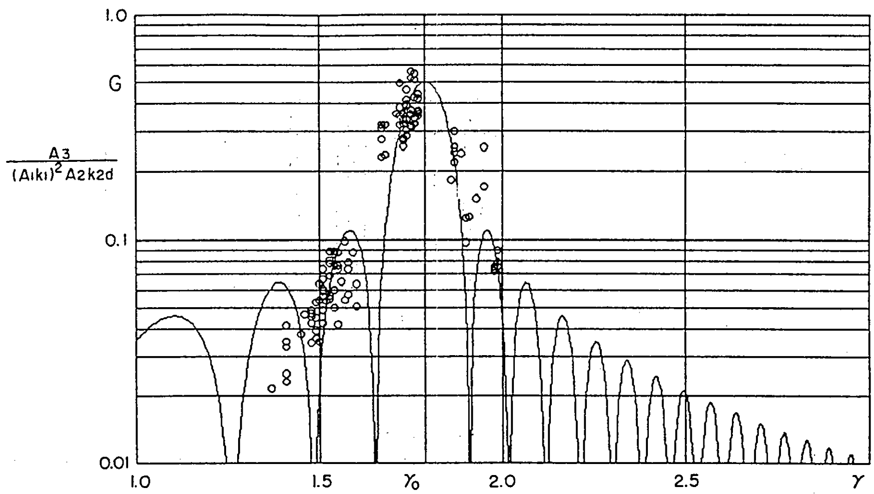


Fig. -2-7
Growth rate of the tertiary resonant waves.
G : the growth rate
γ_θ : γ of the most strong resonance
The solid curves are due to detuning effect.

Table-2-2 Observations of Initial Growth Rate

	G	γ	d
Longuet-Higgins(1962) theoretical value	0.442	1.736	
MacGoldrick et. al. experiment(1966)	0.57	1.78	15 m #
Tomita et.al. experiment(1986)	0.50	1.79	20.25 m

#The distance is converted to the size of our experiment.

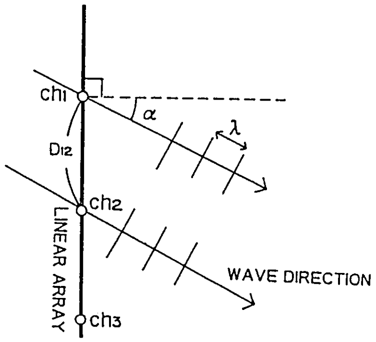


Fig. -2-8
The principle of wave direction measurement.
Ch 1 ~ Ch 3 on the array in a obliquely incident wave

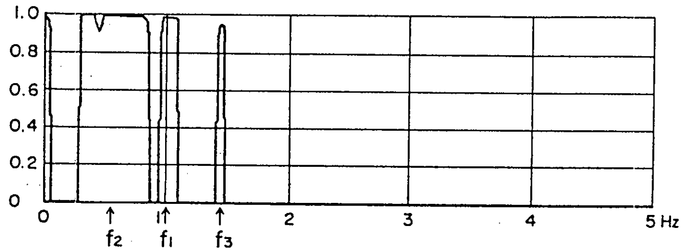


Fig. -2-9
Coherence between wave data at the locations 1 and 3 .
f_1 : 1-st primary wave
f_2 : 2-nd primary wave
f_3 : tertiary resonant wave

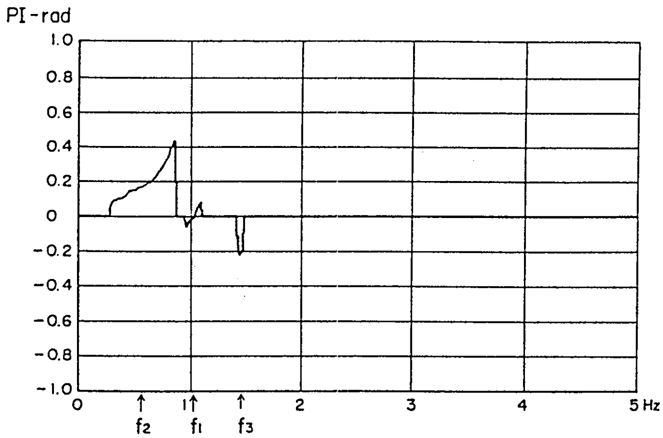
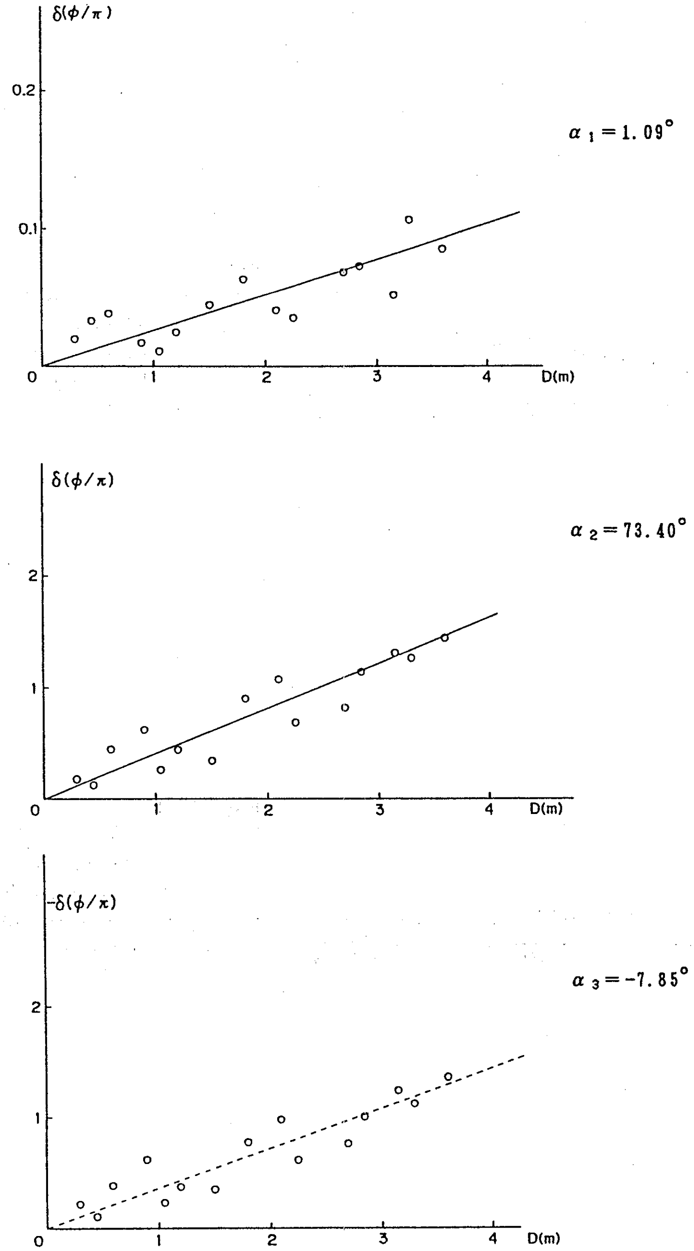


Fig. -2-10
Phase spectrum between wave data at the locations 1 and 3.
f_1 : 1-st primary wave
f_2 : 2-nd primary wave
f_3 : tertiary resonant wave



$$\alpha = \alpha_3 - \alpha_1 = -8.94^\circ$$

Fig. -2-11
Phase differences along the linear array
(a) 1-st primary wave
(b) 2-nd primary wave
(c) tertiary resonant wave

α : angle between the resonant wave and 1-st wave

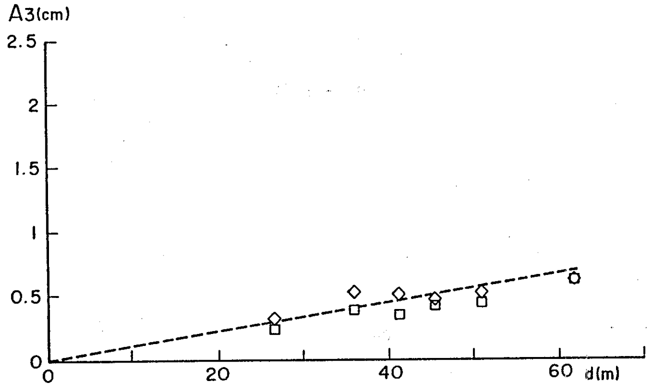


Fig. -2-12
Long term variation of A_3 ($\gamma=1.72$)
------- : Theory (Longuet-Higgins)
□ : Experiment (cm) $A_1=2.29$, $A_2=2.51$
◇ : Experiment (cm) $A_1=2.84$, $A_2=2.50$
Examples of linear growth of resonant waves

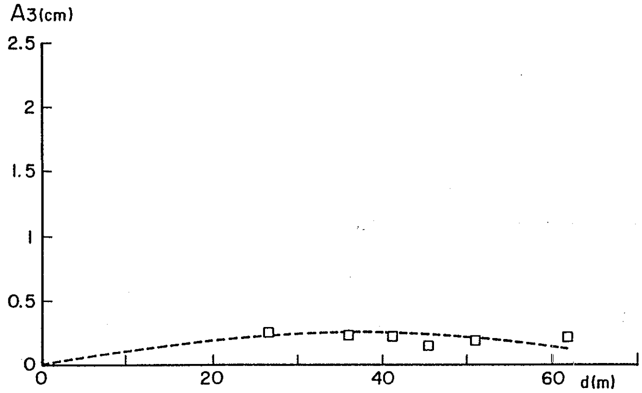


Fig. -2-13
Long term variation of A_3 ($\gamma=1.72$)
------- : Theory (Zakharov)
□ : Experiment (cm) $A_1=4.06$, $A_2=2.51$
Example of weak resonance at the exact (linear) resonance condition

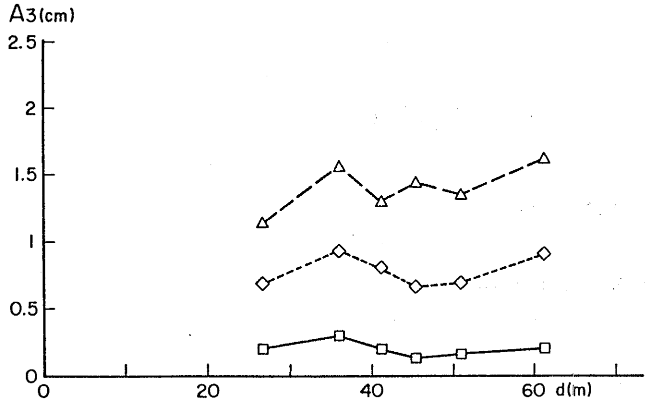


Fig. -2-14
Long term variation of A_3 ($\gamma=1.79$)
□ : Experiment (cm) $A_1=1.80$, $A_2=5.29$
◇ : Experiment (cm) $A_1=2.49$, $A_2=5.03$
△ : Experiment (cm) $A_1=2.84$, $A_2=5.12$
Large resonant wave appears at the off resonance condition.

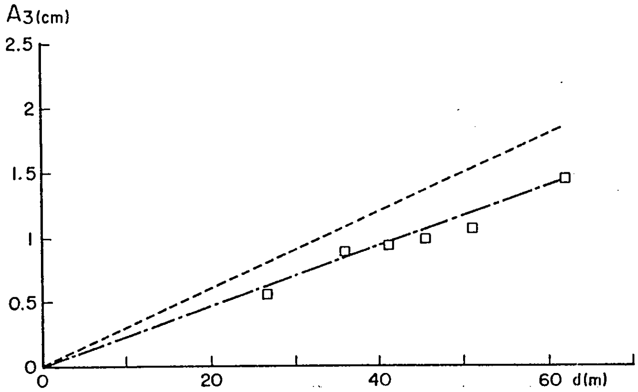


Fig. -2-15
Long term variation of $A_3(\gamma=1.79)$
------- : Theory (Longuet-Higgins)
·-·-·-·-· : Least square fitting
□ : Experiment (cm) $A_1=3.36$, $A_2=2.61$
An example of non-linear resonance

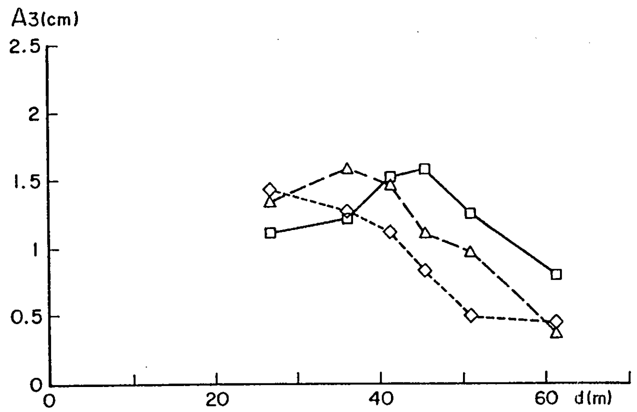


Fig. -2-16
Long term variation of A_3 $(\gamma=1.79)$
□ : Experiment (cm) $A_1=2.91$, $A_2=5.07$
◇ : Experiment (cm) $A_1=3.24$, $A_2=5.28$
△ : Experiment (cm) $A_1=3.44$, $A_2=5.14$
Decreasing of resonant wave amplitudes with fetch

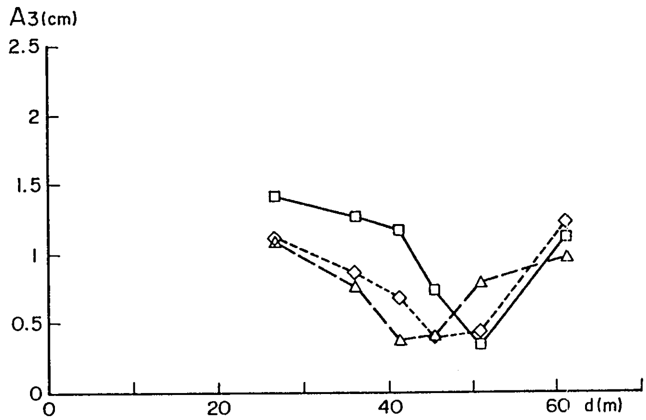


Fig. -2-17
Long term variation of A_3 $(\gamma=1.82)$
□ : Experiment (cm) $A_1=3.63$, $A_2=5.38$
◇ : Experiment (cm) $A_1=3.78$, $A_2=5.40$
△ : Experiment (cm) $A_1=4.15$, $A_2=5.41$
Evidences of recurrence phenomena

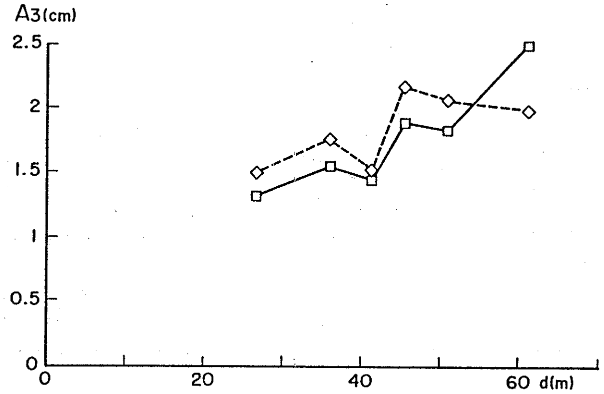


Fig. -2-18
Long term variation of A_3 ($\gamma=1.82$)
□ : Experiment (cm) $A_1=4.76$, $A_2=5.29$
◇ : Experiment (cm) $A_1=5.47$, $A_2=5.35$
The largest amplitudes of resonant waves observed in the experiment.

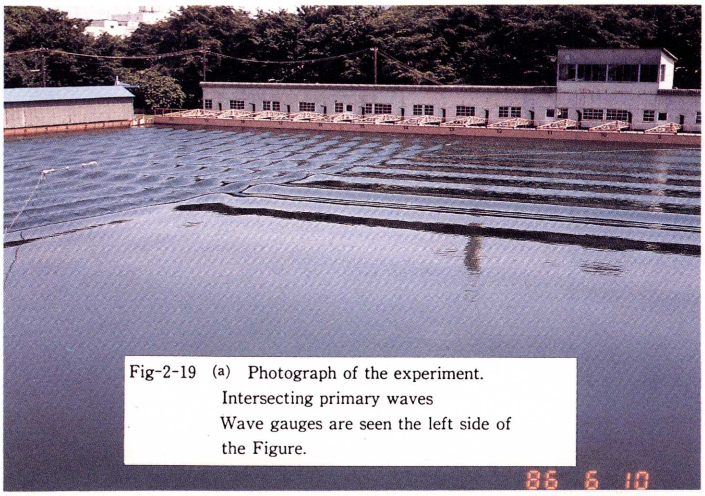


Fig-2-19 (a) Photograph of the experiment.
Intersecting primary waves
Wave gauges are seen the left side of the Figure.

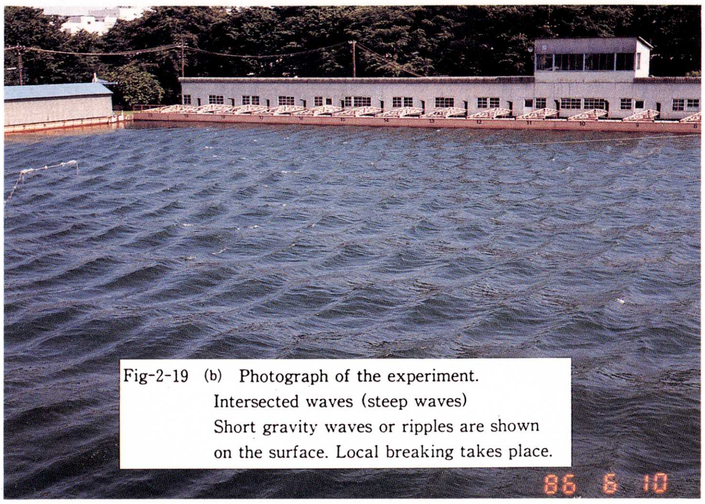


Fig-2-19 (b) Photograph of the experiment.
Intersected waves (steep waves)
Short gravity waves or ripples are shown on the surface. Local breaking takes place.

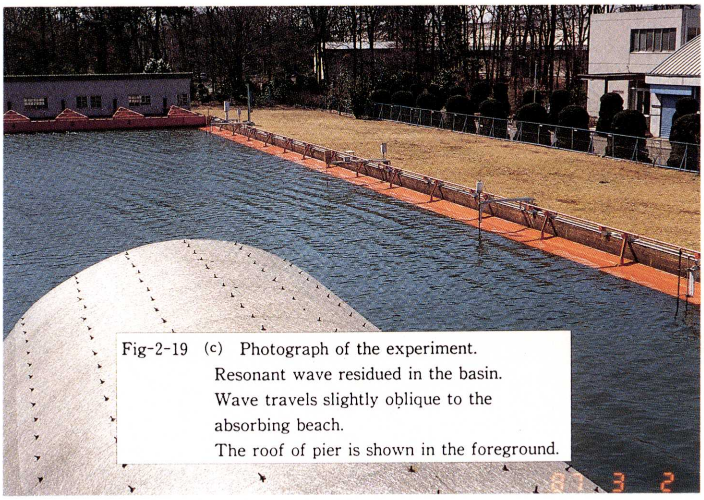


Fig-2-19 (c) Photograph of the experiment.
Resonant wave residued in the basin.
Wave travels slightly oblique to the absorbing beach.
The roof of pier is shown in the foreground.

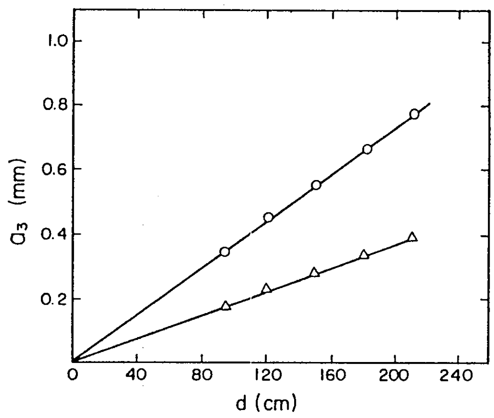


Fig. -3-1
Comparison of the Zakharov theory with the experiments by McGoldrick et. al. (1966) at the short fetch.
○, △ : Experiments (a_2 is one half in the latter)
—— : Theory

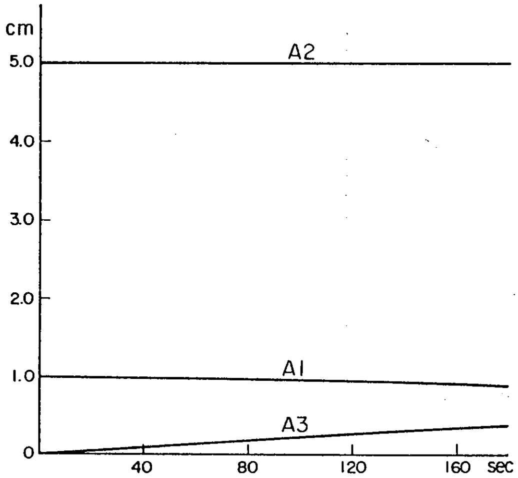


Fig. -3-2 (a)
Solution of the Zakharov equation (γ=1.735)
Initial values : A_1=1.0cm
A_2=5.0cm
A_3=0.0cm
Growth of resonant wave is nearly straight.

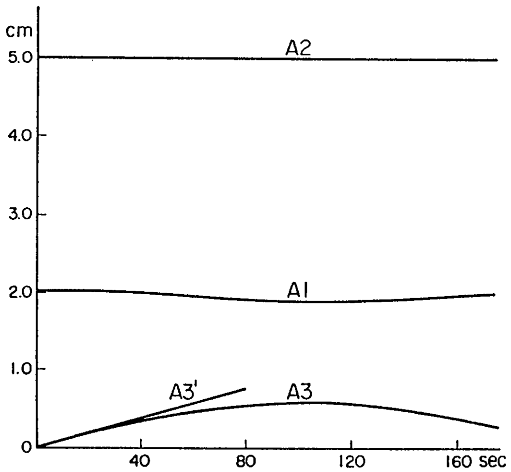


Fig. -3-2 (b)
Solution of the Zakharov equation (γ=1.735)
Initial values : A_1=2.0cm
A_2=5.0cm
A_3=0.0cm
Growth of resonant wave ceases at around 100 sec.
Initial growth rate coinsides with classical one.

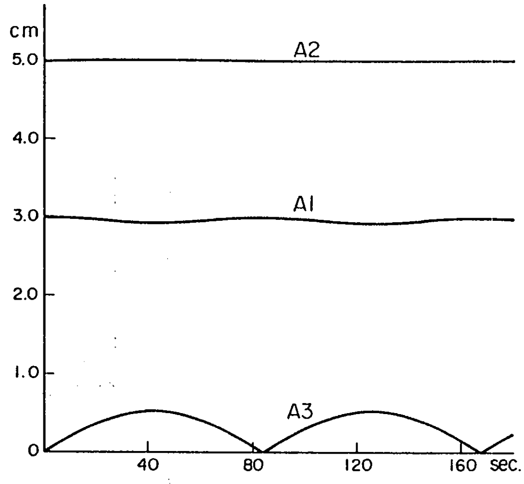


Fig. -3-2 (c)
Solution of the Zakharov equation (γ=1.735)
Initial values : A_1=3.0cm
A_2=5.0cm
A_3=0.0cm
Recurrence phenomena appear.

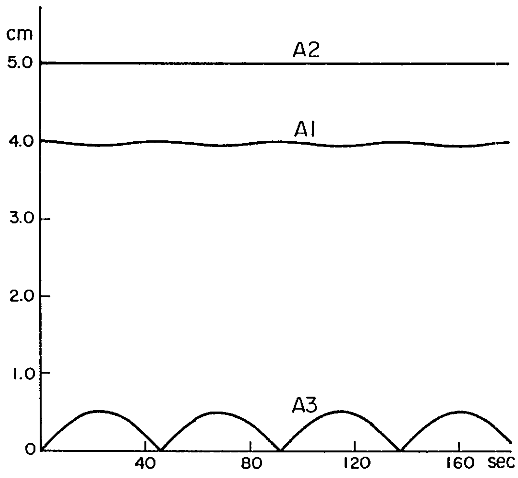


Fig. -3-2 (d)
Solution of the Zakharov equation (γ=1.735)
Initial values : A_1=4.0cm
A_2=5.0cm
A_3=0.0cm
Resonant wave amplitude does not increase proportional to the primary waves.

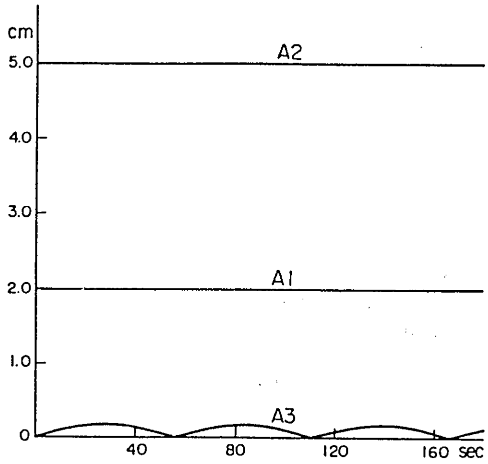


Fig. -3-3 (a)
Solution of the Zakharov equation (γ=1.800)
Initial values : A_1=2.0cm
A_2=5.0cm
A_3=0.0cm

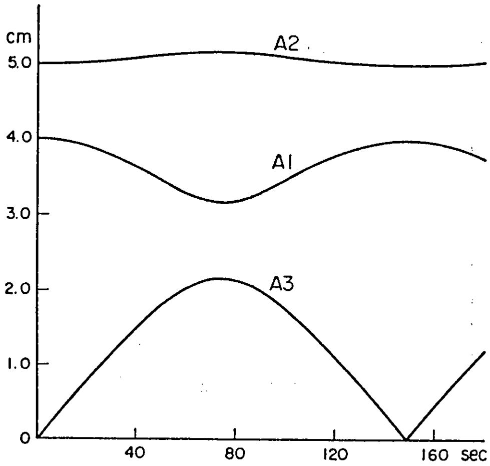


Fig. -3-3 (c)
Solution of the Zakharov equation (γ=1.800)
Initial values : A_1=4.0cm
A_2=5.0cm
A_3=0.0cm
Resonant growth occurs strongly in contrast to the corresponding case in Fig-3-2.

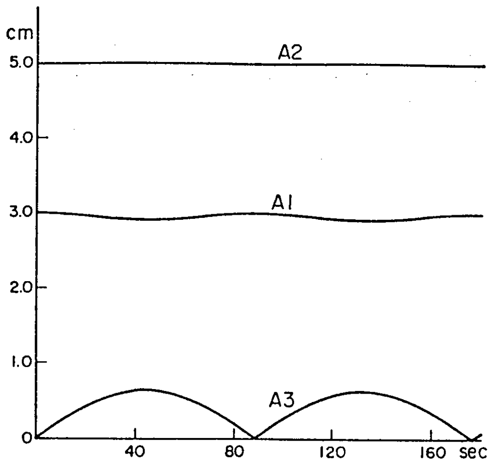


Fig. -3-3 (b)
Solution of the Zakharov equation (γ=1.800)
Initial values : A_1=3.0cm
A_2=5.0cm
A_3=0.0cm

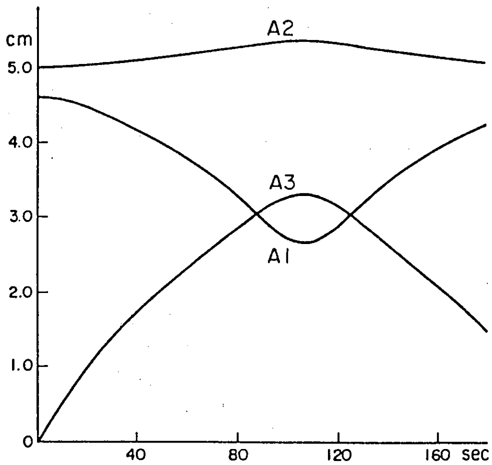


Fig. -3-3 (d)
Solution of the Zakharov equation (γ=1.800)
Initial values : A_1=4.6cm
A_2=5.0cm
A_3=0.0cm
The critical case of interaction.

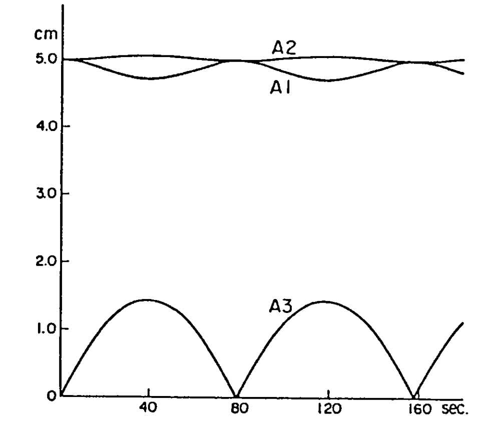


Fig. -3-3 (e)
Solution of the Zakharov equation ($\gamma=1.800$)
Initial values : $A_1=5.0$cm
$A_2=5.0$cm
$A_3=0.0$cm

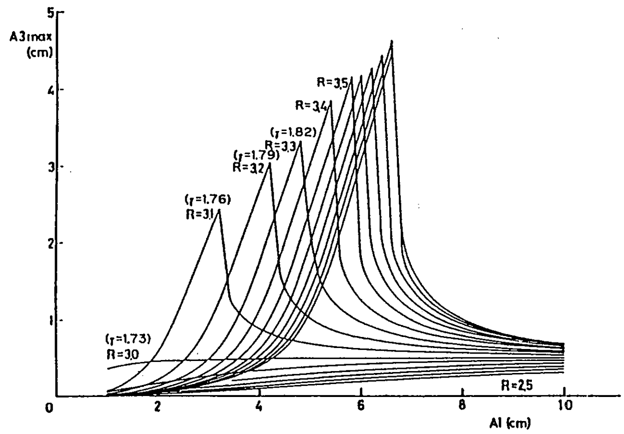


Fig. -3-4
Maximum amplitude A_{3max} v. s. A_1 ($A_2=5$cm)
Dependence of resonant growth of tertiary waves on the primary wave ampritude is shown taking γ as a parameter. There are sharp peaks at off-resonance cases.

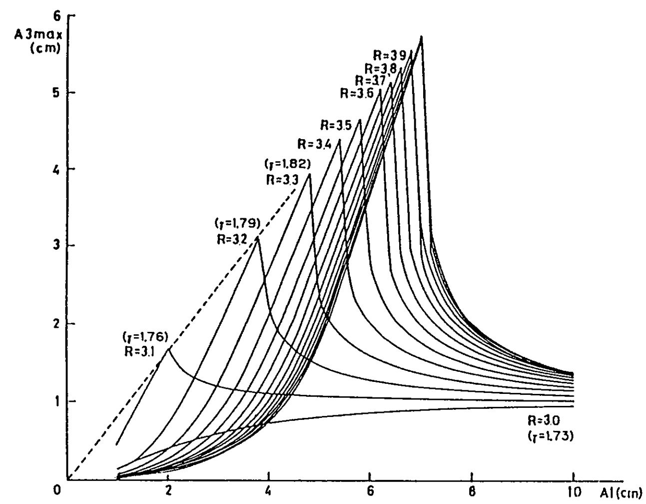


Fig. -3-5
Maximum amplitude A_{3max} v. s. A_1 ($A_2=10$cm)
------- : Limiting line A^{M}_{3max} (3-10)
Upper bounds of resonant wave growth is verified by the numerical experiment.

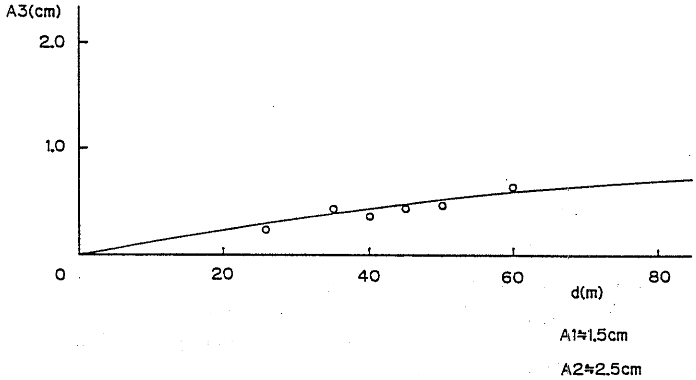


Fig. -3-6 (a)
Evolution of tertiary wave A_3
——— : Theory (Zakharov)
○ : Experiment (cm)
$\gamma=1.72$ (near resonant case)

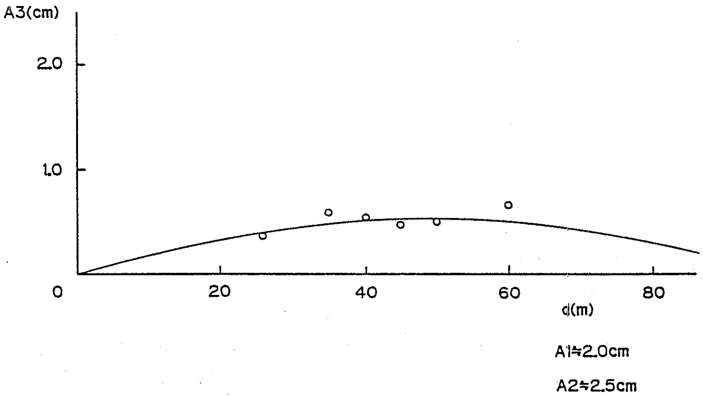


Fig. -3-6 (b)
Evolution of tertiary wave A_3
——— : Theory (Zakharov)
○ : Experiment (cm)
$\gamma=1.72$ (near resonant case)

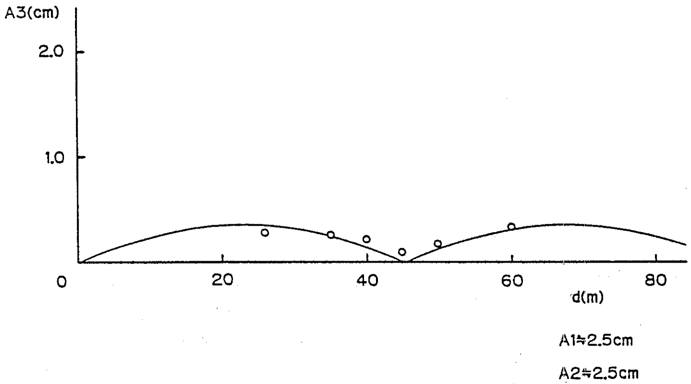


Fig. -3-6 (c)
Evolution of tertiary wave A_3
——— : Theory (Zakharov)
○ : Experiment (cm)
$\gamma=1.72$ (near resonant case)

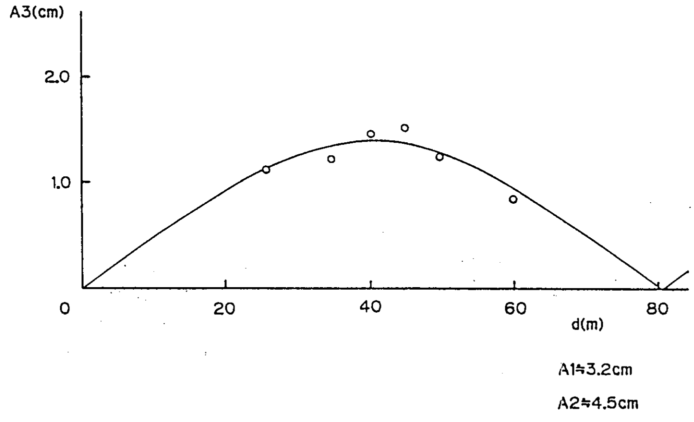


Fig. -3-7 (a)
Evolution of tertiary wave A_3
——— : Theory (Zakharov)
○ : Experiment (cm)
$\gamma=1.79$ (off resonant case)

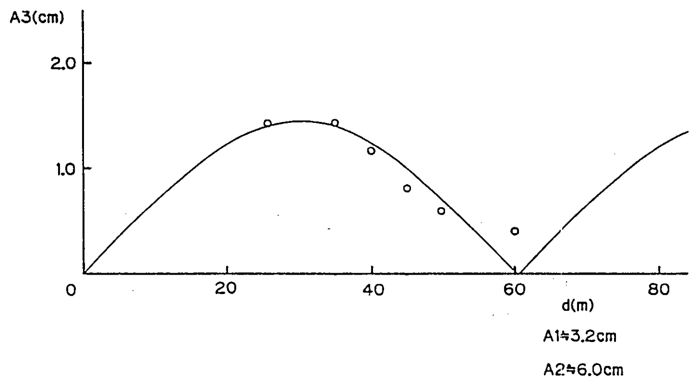


Fig. -3-7 (b)
Evolution of tertiary wave A_3
——— : Theory (Zakharov)
○ : Experiment (cm)
$\gamma=1.79$ (off resonant case)

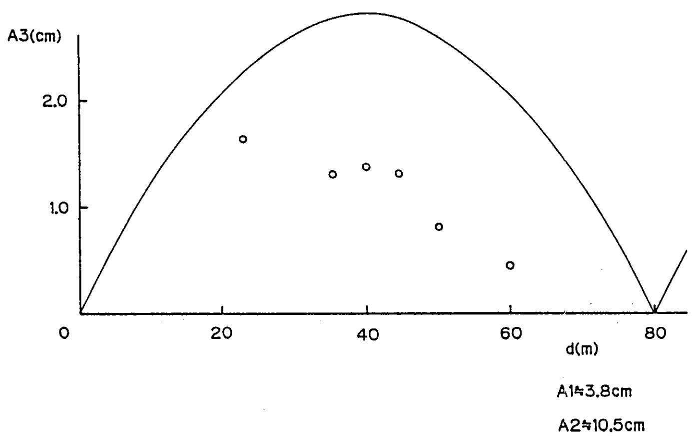


Fig. -3-7 (c)
Evolution of tertiary wave A_3
——— : Theory (Zakharov)
○ : Experiment (cm)
$\gamma=1.79$ (off resonant case)

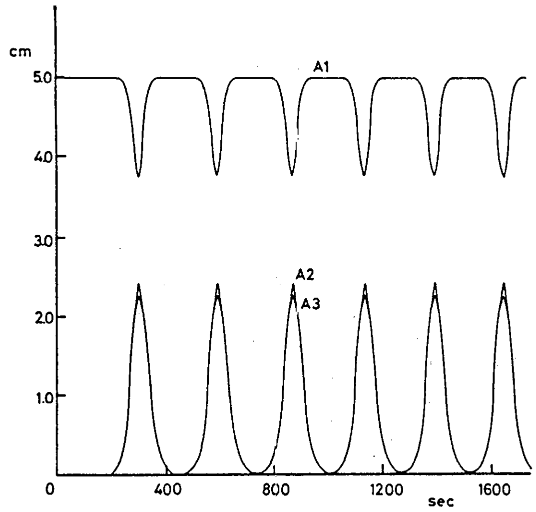


Fig. -3-8
Long-time evolution of a wave train A_1 with its side bands A_2 and A_3
Side band components rise up intermittently.
The recurrence takes place in a very long time instead of disintegration of wave train.

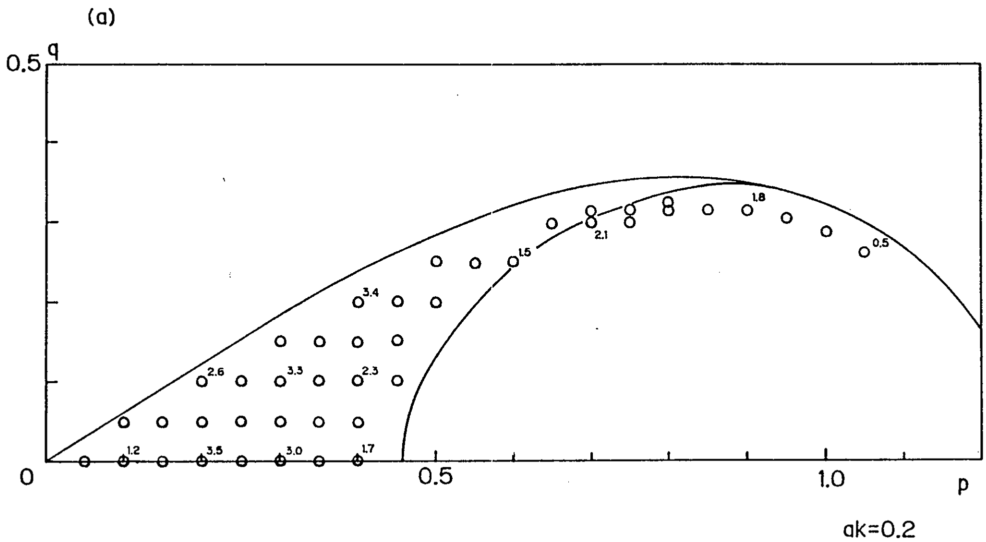


Fig. -3-9 (a)
Domains of instability (wave-number space)
Wave steepness ak=0.2
The solid curve is the instability boundary calculated by McLean (1892).

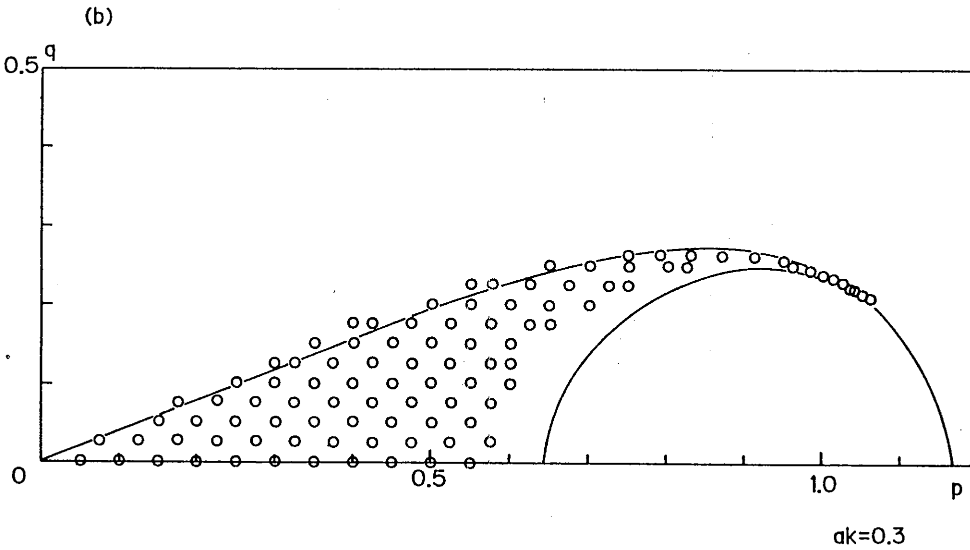


Fig. -3-9 (b)
Domains of instability (wave-number space)
Wave steepness ak=0.3
The solid curve is the instability boundary calculated by McLean (1892).

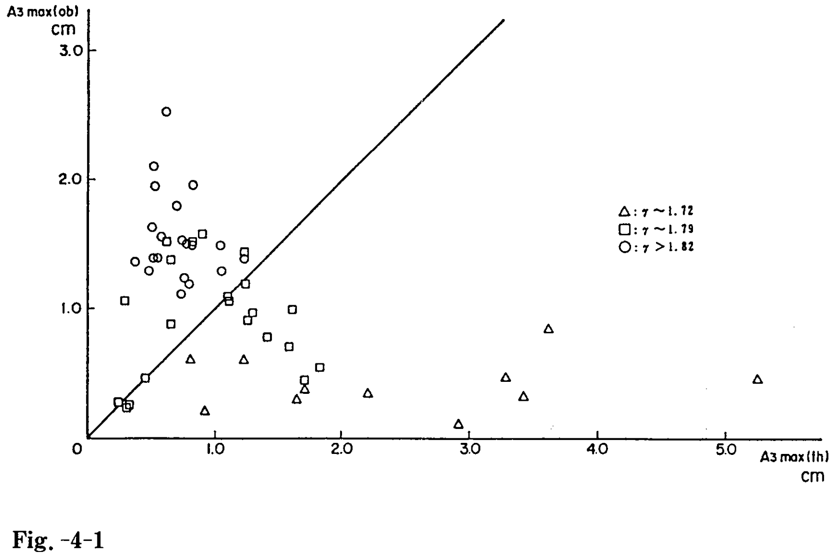


Fig. -4-1
Plot of the observed maximum amplitude of tertiary waves with respect to the theoretical maximum of A_3
A_{3max} (ob) : Experiments
A_{3max} (th) : Theory (Longuet-Higgins)

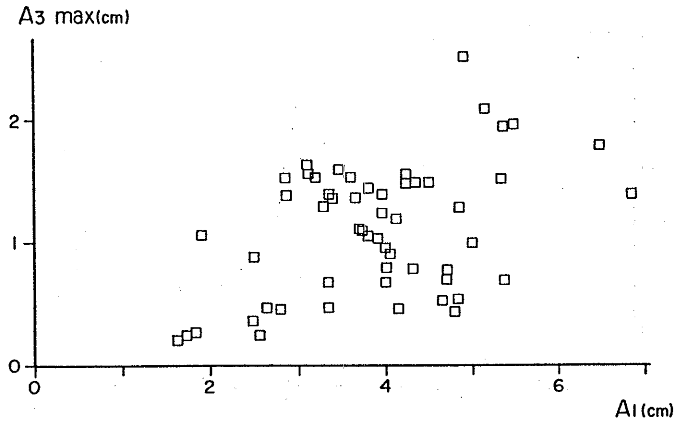


Fig. -4-2
Plot of the observed maximum amplitude A_{3max} with respect to A_1
Various conditions are totally plotted.

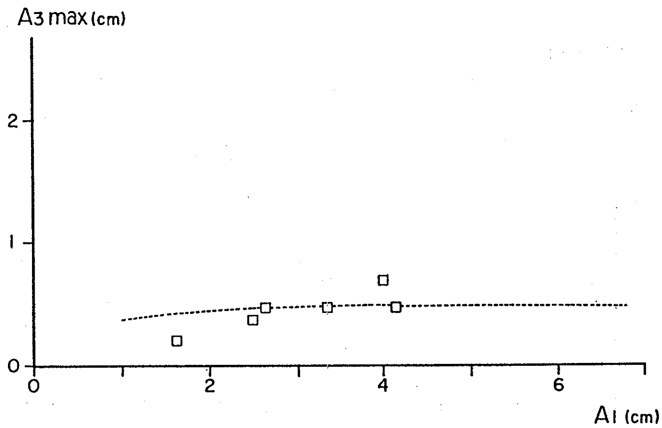


Fig. -4-3
Comparison of observed A_{3max} with Zakharov theory
------- : Theory (Zakharov $\gamma=1.72$)
□ : Experiment (cm) , $A_2=5cm$

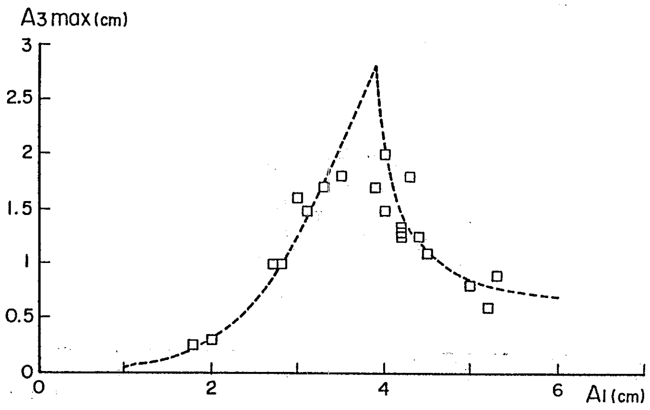


Fig. -4-4
Comparison of observed A_{3max} with Zakharov theory
------- : Theory (Zakharov γ=1.79)
□ : Experiment (cm) , A_2=5cm

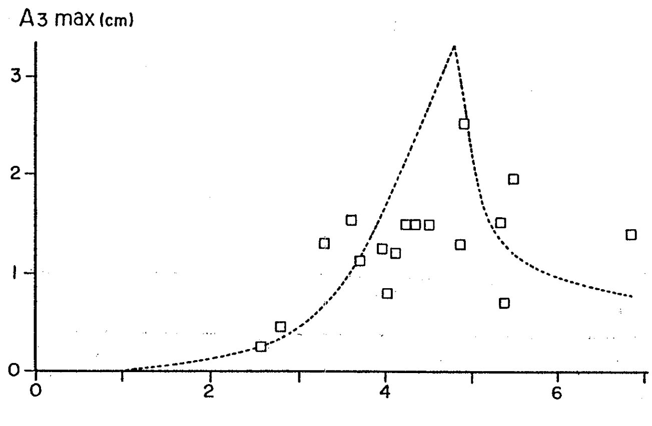


Fig. -4-5
Comparison of observed A_{3max} with Zakharov theory
------- : Theory (Zakharov γ=1.82)
□ : Experiment (cm) , A_2=5cm

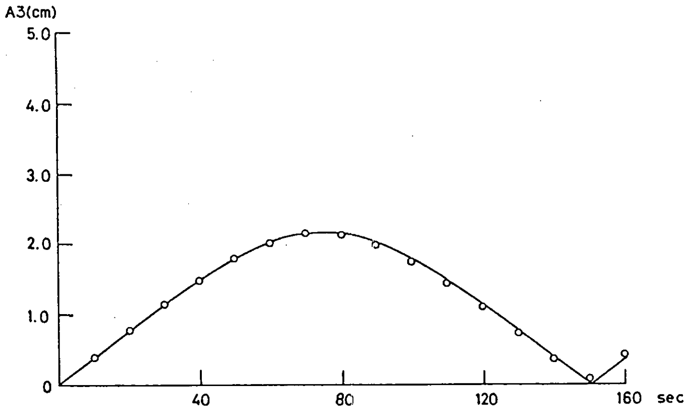


Fig. -A-1

Comparison of analytical solution with the numerical results obtained in Ch3 -an example-

——— : Solution in Appendix IX

○ : Solution in Chapter 3

Initial condition : $A_1 = 4cm$, $A_2 = 5cm$, $A_3 = 0cm$

List of the Related Articles by the Present Author;

(1)Tomita,H. & Sawada,H.(1987) An experimental investigation into non-linear resonant wave interactions in the ship model basin. Proc. IUTAM Sympo.Non-Linear Water Waves.,Springer-Verlag,pp341～348.

(2)Tomita,H.(1987) Etude numerique sur l'interaction resonante entre des vagues d'amplitude finie. La mer,Tome25,pp53～61.

(3)Tomita,H.(1986) On non-linear sea waves and the induced mean flow. J.Oceanogr. Soc. Japan, Vol.42,pp153～160.

(4)Tomita,H.(1985) On non-linear water wave groups and the induced mean flow. Proc.The Ocean Surface Sympo.,D.Reidel Pub.,pp59～64.

(5)Tomita,H.(1988) Ocean Waves and Ship Motions. KOKAI,Vol.96,pp8～16.

(6)Tomita,H.(1988) Wind and wave characteristics in the western North-Pacific Ocean(Part-1). Report of Ship Research Institute,Vol.25,pp. (in printing)

(7)Tomita,H.(1988) Wind and wave characteristics in the western North-Pacific Ocean(Part-2). Report of Ship Research Institute,Vol.25,pp. (in preparation).

(8)Tomita,H.(1985) On mutual effects of large amplitude waves. Proc.46-th Lec.Meeting S.R.I.,pp143～146.

(9)Tomita,H. & Tanizawa,K.(1983) Numerical Investigation into nonlinear water waves by means of the Boundary Element Method. Papers S.R.I., No.69pp1～13.