

Theoretical and Experimental
Investigations of Interaction
among Deep-Water Gravity Waves*

By

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summary

It is well known that most of the energy of sea waves, which causes a lot of damage to ships, off-shore structures and facilities on coasts, is supplied by winds blowing over the ocean. However, if a wind strong enough to generate gravity waves stops, the gravity waves, far from dying out rapidly, will continue to run straight on until they fetch up against something. Once waves have escaped from the wind that made them, they can run for days with very little loss of energy. Therefore, they travel long distance without the influence of winds. Moreover, these wave elements change their properties owing to the mutual interaction during this stage. Accordingly, to understand the nature of sea waves, besides studying the mechanism of wind-wave interaction, it is also imperative to clarify the characteristics of propagation of an individual wave train. In this paper, we deal with the non-linear dynamics of the deep-water gravity waves and apply it to the experiment to interpret the results concerning the mutual interaction among waves.

The contents of each Chapter are as follows. In Chapter 1, we review the basic theory of water waves and formulate the problems from

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the point of view of a singular perturbation method. In the following two Chapters, experimental and numerical studies concerning the particular condition of the resonant wave interactions are described. In Chapter 2, long term evolution of tertiary resonant waves are detected experimentally and the direction of propagation of the resonant wave is also obtained for the first time by aid of the cross-spectral analysis. The purpose of the observations is twofold: to examine quantitatively the evolution of the amplitude modulation and to test the validity of weakly non-linear wave theory (Zakharov equation) for the asymptotic behavior of resonant waves by comparing the predicted and the observed properties of the waves.

In Chapter 3, the Zakharov's integro-differential equation is solved numerically and it is shown that the experimental data agree with the solutions in the case of comparatively small wave steepness. Calculations are also performed to determine the dependence of the maximum amplitude of the resonant wave upon the amplitude of primary waves. In Chapter 4, comparisons of the experimental results with theories are made both for classical and that by Zakharov. It is concluded that the former is insufficient to explain quantitatively the long term evolution of the tertiary resonant wave and that the latter model of non-linear water waves is applicable for describing the propagation of sea waves because of fairly good agreement of the theory with data.

Several theoretical remarks, including the analytical investigation into a particular solution of the discretized Zakharov equation, are offered in Appendices.

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CHAPTER 1 NON-LINEAR DYNAMICS OF WATER WAVES

1. 1 Foreword

It is well known that the work of Stokes titled "On the theory of oscillatory waves" in 1847 is substantially the first study of the non-linear property of water waves. In this pioneering paper, he gave a stationary solution of a train of deep-water gravity waves by aid of the power series expansion with respect to wave steepness. Many important properties of non-linear waves, such as the dispersion relation dependent on amplitude, the existence of highest limit of wave and the drift motion of particles in a wave were shown in his work. Besides the above mentioned theory, the Trochoidal wave, an exact particular solution of water wave, found by Gerstner(1809), had been applied in the field of naval architecture for a long time. These basic solutions are the most important ones in the non-linear water wave theory.

On the other hand, the researches concerning the description of ocean waves have been developed in a somewhat different manner. In this field, the subject is divided into two main parts. One is to investigate the mechanisms of wave generation by wind. The other is to describe the actual configuration of ocean surface properly.

In this paper, we deal mainly with the latter problem. The study of the scientific description of sea waves was started at the beginning of 1950s with the work of Pierson(1952) who introduced the concepts of stochastic processes and of spectrum to oceanography. His investigation for wave forecasting has been developed considerably by aid of electronic computers. However, from the theoretical point of view, there is enough ground for controversy in his method. Pierson, Neumann & James (1955) assumed that the fluctuation of the ocean surface is composed of many infinitesimal wave trains which travel independently to each other in their own directions. According to this assumption, the spectrum of sea wave is recognized as a distribution function of the energy of component waves. On the contrary, the stochastic variation of surface displacement, its velocity or acceleration satisfies the Gaussian distribution and the moments can be determined by the spectrum. So far as we admit the linear wave theory, there would be no problem conceptually.

Once we draw attention to the non-linear properties of water waves and consider them in the framework of the PNJ method, most of the concepts would become ambiguous. However, no one could have extended the theory to contain the non-linear characteristics of waves in the ages of

1950s, because the theory of non-linear waves had been no more improved than that established in 19th century. To overcome this difficulty, there appeared many papers concerning the non-linear theory with regard to multiple component wave system since 1960s. We mention some of them, in relation to this paper such as Tick(1959) and Hamada(1966), which are the second order theory for random wave field. Huang & Tung(1976), Weber & Barrick(1977), Barrick & Weber(1977), Masuda, Mitsuyasu & Kuo(1979) and Mitsuyasu, Kuo & Masuda(1979) dealt with the third order random wave field although they did not take the energy transfer among component waves into account except for the change of wave velocity. The last one involves the experimental verification in a wind-wave flume. In earlier, Phillips(1960) proposed the theory for accounting the energy transfer between wave components however his mathematical formulation contained a singular property and did not offer the solution describing the long-time evolution of resonant waves. Benney(1962) gave the equations which describe the long-time behavior of four waves for the first time.

Zakharov(1968) derived the equation governing the mutual interaction among deep-water gravity waves of arbitrary number of components in the most purely theoretical point of view. Stiassnie & Shemer(1984) rederived it by somewhat elementary method with using Fourier transform technique. They are most closely related ones to the present paper. In this Chapter, we reexamine those works and discuss the non-linear dynamics of water waves in the unified point of view. Some precise study concerning the characteristics of Zakharov equation containing the numerical and analytical solution will be discussed in Chapter 3 and in Appendices.

In addition, we also mention the book "The Dynamics of the Upper Ocean" written by Phillips(1977) as the most excellent description and the basic results of sea waves. The simple and fine explanations are referred in the articles written by Nagata(1970) and Taira(1975).

1. 2 Basic Equations

In this Chapter, we assume in regard to hydrodynamic natures of water waves that the viscosity is neglected (perfect fluid), and that the motion is irrotational and the compressibility of the fluid is neglected. Capillarity and air motion above the surface of fluid are not taken into account. The density of water is assumed not to change temporally and spatially.

We deal the present problem as, in three dimensional space, that is, two dimensional sufficiently large horizontal surface which is uni-

form and isotropic. The depth of sea is infinite. We also assume that the amplitude of the wave is small but finite.

From the assumption of irrotational motion, there exists the velocity potential ϕ in the fluid. By the assumption of incompressibility, the equation of continuity is satisfied

$$\nabla^2 \phi = 0 \quad (1-1)$$

in the interior of the fluid. Here, we take the coordinate system as x - and y -axes in horizontal and z -axis in the vertical upwards direction respectively. At the fluid surface ($z = \eta$) the kinematic boundary condition

$$\frac{\partial \eta}{\partial t} + \nabla_h \phi \nabla_h \eta = \frac{\partial \phi}{\partial z} \quad (1-2)$$

and the dynamic boundary condition

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \nabla \phi \nabla \phi = -g \eta \quad (1-3)$$

are satisfied. Where, η denotes the displacement of the surface and g represents the acceleration due to the gravity. The operator ∇_h means the horizontal components of gradient operator ∇ . From the assumption, the density of water is constant so that it does not appear in these equations. The difficulty of the problems on water waves lies on the fact that the above equations (1-2) and (1-3) are both non-linear and the form of the boundary η is not determined ab initio but is an unknown variable. Finally, from the assumption in the limit $z \rightarrow -\infty$,

$$\nabla \phi \rightarrow 0 \quad (1-4)$$

is required.

1. 3 Some Aspects on Classical Theory

On the basis of the general theory in hydrodynamics, we restrict ourselves to the problem of non-linear resonant wave interaction. Phillips(1960) discovered that in the third approximation, it is possible for a transfer of energy to take place from three primary waves (of wave-numbers k_1 , k_2 and k_4) to a fourth wave (of wave-number k_3) in such a way that the amplitude of the fourth wave increases linearly with

time. Thus, although the fourth wave amplitude at first is very small (being of the third order) it may grow in time so as to be comparable with the three primary waves. The condition for this is that the wave-numbers k_1, k_2, k_3, k_4 and frequencies $\omega_1, \omega_2, \omega_3, \omega_4$ each satisfy the dispersion relation:

$$\omega_i^2 = g |k_i| \quad (i=1, 2, 3, 4), \quad (1-5)$$

and that

$$k_1 \pm k_2 \pm k_3 \pm k_4 = 0, \quad \omega_1 \pm \omega_2 \pm \omega_3 \pm \omega_4 = 0, \quad (1-6)$$

with the same combination of signs in each case.

At first, we explain briefly the theoretical results obtained by the direct use of a perturbation technique (REGULAR PERTURBATION) to the basic equations. Longuet-Higgins (1962) has analysed this problem in the case that $k_1 = k_4, \omega_1 = \omega_4$, the condition (1-6) turns out to be

$$2k_1 - k_2 = k_3, \quad 2\omega_1 - \omega_2 = \omega_3. \quad (1-7)$$

Phillips (1960) showed that in the case that resonance condition (1-7) is satisfied, wave-number k_1, k_2 and k_3 should be correlated each other as shown in Fig-1-1. In the special condition that $k_1 \perp k_2$, $\gamma = \omega_1 / \omega_2$ would be $\gamma = \gamma_0 = 1.736 \dots$.

The velocity potential ϕ and surface displacement η are assumed to be expressed in expanded series such that

$$\begin{aligned} \phi = & (\alpha \phi_{10} + \beta \phi_{01}) + (\alpha^2 \phi_{20} + \alpha \beta \phi_{11} + \beta^2 \phi_{02}) + \\ & + (\alpha^3 \phi_{30} + \alpha^2 \beta \phi_{21} + \alpha \beta^2 \phi_{12} + \beta^3 \phi_{03}) + \dots \quad (1-8-1) \end{aligned}$$

and

$$\begin{aligned} \eta = & (\alpha \eta_{10} + \beta \eta_{01}) + (\alpha^2 \eta_{20} + \alpha \beta \eta_{11} + \beta^2 \eta_{02}) + \dots \\ & + (\alpha^3 \eta_{30} + \alpha^2 \beta \eta_{21} + \alpha \beta^2 \eta_{12} + \beta^3 \eta_{03}) + \dots \quad (1-8-2) \end{aligned}$$

with α and β being independent small quantities representing the wave steepness of each wave. Substituting (1-8-1) and (1-8-2) to the basic equations (1-2) and (1-3), the calculations were carried out up to the third order terms. We only pay attention to the term ϕ_{21} , because it represents the tertiary resonant wave to be considered here. Longuet-Higgins & Smith (1966) gave solution ϕ_{21} at $z=0$ as

$$\phi_{21} = -\frac{K}{g \delta k} \sin(\delta k x) \sin\{(k_0 + \delta k)x - \omega_3 t\}, \quad (1-9)$$

under a slightly extended conditions that

$$2k_1 - k_2 = k_3, \quad 2\omega_1 - \omega_2 \sim \omega_3. \quad (1-10)$$

In (1-10), equality might not be satisfied strictly for frequencies. In the equation (1-9), K is the growth rate and expressed as

$$K = (a_1 k_1)^2 a_2 k_2 g^2 \omega_3^{-1} G,$$

with non-dimensional coefficient G . k_0 equals to ω_0^2/g where ω_0 is defined as $\omega_0 = 2\omega_1 - \omega_2$ and $2\delta k = k_3 - k_0$. δk and $\delta\gamma = \gamma - \gamma_0$ are correlated as

$$\frac{2\delta k}{k_3} = -\left(\frac{4}{2\gamma_0 - 1} - \frac{8\gamma_0^3}{4\gamma_0^4 + 1}\right)\delta\gamma. \quad (1-11)$$

From the form of (1-9), we can recognize that amplitude of tertiary wave varies slowly with x when $\delta\gamma \neq 0$. If $\delta\gamma = 0$, the solution ϕ_{21} in (1-9) appears to be infinite, however in such a limiting case, it reduces to

$$\phi_{21} \rightarrow -\frac{Kx}{g} \frac{\sin \delta k x}{\delta k x} \sim \frac{Kx}{g}. \quad (1-12)$$

Thus, tertiary wave grows linearly with x . Transforming it to the wave amplitude a , the maximum amplitude a_{3M} to be realized by tertiary wave is obtained as

$$a_{3M} = \frac{(a_1 k_1)^2 a_2 k_2}{|\delta k|} G. \quad (1-13)$$

The constant G is given by Longuet-Higgins(1962) as 0.442. In order to express a_{3M} by the explicit function of $\delta\gamma$, we eliminate δk in (1-9) by using (1-11), we have a classical approximation

$$\frac{a_{3H}}{a_2} \approx 0.491 \frac{(a_1 k_1)^2}{|\delta \gamma|} . \quad (1-14)$$

1.4 Expansion Procedure of the Solution

In this section, we derive the equation which governs the interaction among components of gravity wave system. The method of derivation is essentially different from the classical one as explained in § 1.3 and is applicable to developing stage of non-linear interactions. In order to consider generally the two-dimensional multiple component wave system, the velocity potential ϕ and sea surface displacement η in the basic equations (1-1) ~ (1-4) are expressed as spatial Fourier series of the forms,

$$\phi(\mathbf{r}, z, t) = \sum_{\mathbf{k}} A(\mathbf{k}, z, t) \exp(i\mathbf{k}\mathbf{r}) \quad (1-15)$$

and

$$\eta(\mathbf{r}, t) = \sum_{\mathbf{k}} B(\mathbf{k}, t) \exp(i\mathbf{k}\mathbf{r}) . \quad (1-16)$$

From the pure mathematical point of view, Fourier integral or Fourier-Stieltjes integral representation must be used, but according to Weber & Barrick(1977), in the case of the assumption that the horizontal area considered here is finite though sufficiently larger than typical wavelength, equations (1-15), (1-16) are hold good. As ϕ satisfies the conditions (1-1) and (1-4), velocity potential ϕ has the form

$$\phi(\mathbf{r}, z, t) = \sum_{\mathbf{k}} A(\mathbf{k}, t) \exp(kz + i\mathbf{k}\mathbf{r}) . \quad (1-17)$$

In the process from now on, the several points explained in the following subsections should be considered carefully;

1.4.1 Treatment of the boundary conditions on unfixed surface $z = \eta$

When we treat the basic equations (1-2) and (1-3) on the surface $z = \eta$, all the terms containing derivatives of ϕ are proportional to $\exp(k\eta)$. We assume the wave steepness $k\eta \ll 1$ and use the Taylor expansion

$$\exp(k\eta) = 1 + k\eta + (1/2)k^2\eta^2 + (1/6)k^3\eta^3 + \dots$$

For example, ϕ_t is calculated in the following way.

First, we differentiate (1-17) with respect to t and insert η in place of z . Next, we use the Taylor expansion of the exponential function above and substitute the expression (1-16) into the powers of η . We finally obtain in the form of spatial Fourier series as

$$\frac{\partial \phi}{\partial t} = F_1 + F_2 + F_3 + \dots$$

Where, F_n ($n=1, 2, 3, \dots$) represents the n -th order quantities as

$$F_1 = \Sigma_k A_t(\mathbf{k}) \exp(i\mathbf{k}\mathbf{r}),$$

$$F_2 = \Sigma_k \exp(i\mathbf{k}\mathbf{r}) [\Sigma_{k_1} k_1 A_t(\mathbf{k}_1) B(\mathbf{k}-\mathbf{k}_1)]$$

$$F_3 = \frac{1}{2} \Sigma_k \exp(i\mathbf{k}\mathbf{r}) [\Sigma_{k_1} B(\mathbf{k}-\mathbf{k}_1) \{ \Sigma_{k_2} k_2^2 A_t(\mathbf{k}_2) B(\mathbf{k}_1-\mathbf{k}_2) \}],$$

and

.....

Calculating $\nabla \phi$ in the similar manner, the results are substituted into (1-2) and (1-3). Utilizing the orthogonality property of Fourier series, we can transform the basic equations to the simultaneous differential equation with respect to A and B . We can finally obtain the results up to the third order of A and B as follows

$$B_t(\mathbf{k}) - k A(\mathbf{k}) = \Sigma_{k_1} \{ k_1 \cdot (\mathbf{k}-\mathbf{k}_1) + k_1^2 \} A(\mathbf{k}_1)$$

$$B(\mathbf{k}-\mathbf{k}_1) + \Sigma_{k_1} B(\mathbf{k}-\mathbf{k}_1) \Sigma_{k_2} \{ k_2 k_2 \cdot (\mathbf{k}_1-\mathbf{k}_2) +$$

$$\frac{1}{2} k_2^3 \} A(\mathbf{k}_2) B(\mathbf{k}_1-\mathbf{k}_2) \quad (1-18)$$

and

$$\begin{aligned}
A_t(\mathbf{k}) + g B(\mathbf{k}) &= \frac{1}{2} \Sigma_{k_1} \{ \mathbf{k}_1 \cdot (\mathbf{k} - \mathbf{k}_1) - k_1 | \mathbf{k} - \mathbf{k}_1 | \} \\
A(\mathbf{k}_1) A(\mathbf{k} - \mathbf{k}_1) - \Sigma_{k_1} k_1 A_t(\mathbf{k}_1) B(\mathbf{k} - \mathbf{k}_1) - \\
\frac{1}{2} \Sigma_{k_1} B(\mathbf{k} - \mathbf{k}_1) \Sigma_{k_2} k_2^2 A_t(\mathbf{k}_2) B(\mathbf{k}_1 - \mathbf{k}_2) + \\
\Sigma_{k_1} B(\mathbf{k} - \mathbf{k}_1) \Sigma_{k_2} \{ k_2 k_2 \cdot (\mathbf{k}_1 - \mathbf{k}_2) - k_2^2 | \mathbf{k}_1 - \mathbf{k}_2 | \} \\
A(\mathbf{k}_2) A(\mathbf{k}_1 - \mathbf{k}_2) . \qquad \qquad \qquad (1 - 19 - 1)
\end{aligned}$$

Here, suffix t means the time derivatives and from now on, we use the expression $A(\mathbf{k})$ instead of $A(\mathbf{k}, t)$ omitting independent variable t . Except for Phillips(1960), Zakharov(1968) and Stiassnie & Shemer(1984), theories by the other authors were restricted that A, B are periodic functions so that the equations were reduced merely to algebraic relations (in fact, setting $A, B \propto \exp(-i\omega t)$, we could show that equations (1-18) and (1-19-1) reduce to those in Weber & Barrick(1977) after some simple algebraic manipulation).

1. 4. 2 Transforming the equations to apply the singular perturbation method

In order to arrange the equations to apply the SINGULAR PERTURBATION METHOD, time derivative A_t in the right-hand side of (1-19-1) has to be eliminated. At first, we neglect terms higher than second order and we have

$$A_t(\mathbf{k}) \cong -g B(\mathbf{k})$$

in the first order. Substituting this into (1-19) iteratively, we obtain the second order approximation as,

$$\begin{aligned}
A_t(\mathbf{k}) \cong -g B(\mathbf{k}) + \frac{1}{2} \Sigma_{k_1} k_1 \cdot (\mathbf{k} - \mathbf{k}_1) A(\mathbf{k}_1) \\
A(\mathbf{k} - \mathbf{k}_1) + \Sigma_{k_1} g k_1 B(\mathbf{k}_1) B(\mathbf{k} - \mathbf{k}_1) .
\end{aligned}$$

Using them in (1-19) again, up to the third order it is transformed into

$$\begin{aligned}
A_t(\mathbf{k}) + g B(\mathbf{k}) &= \frac{1}{2} \Sigma_{k_1} \{ \mathbf{k}_1 \cdot (\mathbf{k} - \mathbf{k}_1) - k_1 |\mathbf{k} - \mathbf{k}_1| \} \\
&A(\mathbf{k}_1) A(\mathbf{k} - \mathbf{k}_1) + \Sigma_{k_1} g k_1 B(\mathbf{k}_1) B(\mathbf{k} - \mathbf{k}_1) - \\
&\Sigma_{k_1} B(\mathbf{k} - \mathbf{k}_1) \Sigma_{k_2} g k_2 (k_1 - \frac{1}{2} k_2) B(\mathbf{k}_2) B(\mathbf{k}_1 - \mathbf{k}_2) \\
&- \Sigma_{k_1} B(\mathbf{k} - \mathbf{k}_1) \Sigma_{k_2} (\frac{1}{2} k_1 - k_2) \{ \mathbf{k}_2 \cdot (\mathbf{k}_1 - \mathbf{k}_2) - \\
&k_2 |\mathbf{k}_1 - \mathbf{k}_2| \} A(\mathbf{k}_2) A(\mathbf{k}_1 - \mathbf{k}_2). \quad (1-19-2)
\end{aligned}$$

The combination (1-18) and (1-19-2) reduces to the equations of harmonic oscillation in the limit $A, B \rightarrow 0$.

1. 4. 3 Technique for eliminating the variable A or B with the consideration that ϕ and η are real quantities

A and B are the Fourier coefficients of the velocity potential ϕ and the surface displacement η . As ϕ and η are real numbers, A and B must be complex numbers whose dependence on \mathbf{k} have the anti-symmetric nature

$$A(\mathbf{k}) = A^*(-\mathbf{k}) \quad (1-20-1)$$

and

$$B(\mathbf{k}) = B^*(-\mathbf{k}). \quad (1-20-2)$$

Where, A^* is a complex conjugate of A. Thus, we can introduce such a complex variable Z that

$$i \alpha_k A(\mathbf{k}) = Z(\mathbf{k}) - Z^*(-\mathbf{k}) \quad (1-21-1)$$

$$\beta_k B(\mathbf{k}) = Z(\mathbf{k}) + Z^*(-\mathbf{k}) \quad (1-21-2)$$

for the reason that the relations (1-20-1) and (1-20-2) are satisfied automatically. In these relations, α_k and β_k are the real constants dependent only upon the magnitude of the wave-number k.

If we execute the transformation (1-21), we can deal two unknowns A and B as in one unknown Z formally. The resultant equation of Z again reduces to that of harmonic oscillation only if the constants α_k and β_k satisfy the relation

$$k \alpha_k^{-2} = g \beta_k^{-2} \quad (1-22)$$

In this paper, according to Stiassnie & Shemer(1984),

$$\alpha_k^2 = 2 (k/g)^{1/2}, \quad \beta_k^2 = 2 (g/k)^{1/2} \quad (1-23)$$

is adopted.

The equations (1-18) and (1-19-2) are transformed by means of (1-21). If Z_t^* is eliminated in these equations, the linear part of Z^* also vanishes owing to (1-22). Thus, the equation with respect to Z is obtained in the following such that

$$i Z_t - (gk)^{1/2} Z = J(k, Z)$$

where, $J(k, Z)$ is yielded by $i \alpha^{-1}$ times of the right-hand side of (1-18) minus β^{-1} times of the right-hand side of (1-19-2).

Explicit form of the equation is

$$\begin{aligned} i Z_{kt} - \omega_k Z_k = & \sum_{k_1, k_2} H^{(1)}(k, k_1, k_2) Z_{k_1} Z_{k_2} + \sum_{k_1, k_2} H^{(2)}(k, \\ & k_1, k_2) Z_{k_1} Z_{k_2}^* + \sum_{k_1, k_2} H^{(3)}(k, k_1, k_2) \\ & Z_{k_1}^* Z_{k_2}^* + \sum_{k_1, k_2, k_3} F^{(1)}(k, k_1, k_2, k_3) Z_{k_1} Z_{k_2} \\ & Z_{k_3} + \sum_{k_1, k_2, k_3} F^{(2)}(k, k_1, k_2, k_3) Z_{k_1}^* Z_{k_2} Z_{k_3} + \\ & \sum_{k_1, k_2, k_3} F^{(3)}(k, k_1, k_2, k_3) Z_{k_1}^* Z_{k_2}^* Z_{k_3} + \\ & \sum_{k_1, k_2, k_3} F^{(4)}(k, k_1, k_2, k_3) Z_{k_1}^* Z_{k_2}^* Z_{k_3}^* \end{aligned} \quad (1-24)$$

Where, ω_k is angular frequency given by $\omega_k = (gk)^{1/2}$ which is the dispersion relation of deep-water gravity waves. Equation (1-24) is the MODE COUPLING EQUATION to describe the propagation of finite amplitude water waves discussed in this paper. The concrete expression of the coefficients $H^{(n)}(k, k_1, k_2)$ and $F^{(n)}(k, k_1, k_2, k_3)$ are presented in Appendix I. By use of the complex amplitude Z , surface elevation η is represented as

$$\eta(\mathbf{r}, t) = \frac{1}{2} \sum_{\mathbf{k}} \beta_{\mathbf{k}}^{-1} [Z(\mathbf{k}, t) + Z^*(-\mathbf{k}, t)] \exp(i \mathbf{k} \cdot \mathbf{r}). \quad (1-25)$$

1.5 Perturbation Method and Zakharov Theory

In this section we apply the singular perturbation method to analyse non-linear equation like (1-24) in contrast to the regular perturbation method used in § 1.3. As discussed briefly in § 1.3, the application of the regular perturbation method to non-linear equation results in the solution infinitely increasing with time t . This fact means that the method is not suitable to express the long-time variation of the solutions. Therefore, to avoid such a difficulty and to obtain the long-time evolution of solution, we adopt here the MULTIPLE SCALE METHOD, a sort of the singular perturbation method. The essence of the method lies on the technique introducing the slowly varying independent variables. We execute this procedure somewhat more systematically than Zakharov(1968) or Stiassnie & Shemer(1984). This method is applicable only to the non-linear equations of the form discussed in § 1.4.2 of the preceding section (Bogoliubov & Mitropolskii(1965) called them quasi-linear equation).

Now, we introduce a small parameter ε and expand Z as

$$Z = \varepsilon Z^{(1)} + \varepsilon^2 Z^{(2)} + \varepsilon^3 Z^{(3)} + \dots \quad (1-26)$$

Furthermore, we introduce a group of independent variables $T_n = \varepsilon^n t$ instead of t . Then, Z is regarded as the function not only of t but also of T_n ($n=1, 2, 3, \dots, T_0 = t$). So, the equation (1-24) becomes a partial differential equation. Differential operator is also expanded as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \varepsilon^3 \frac{\partial}{\partial T_3} + \dots \quad (1-27)$$

We substitute (1-26) and (1-27) into (1-24) and rearrange it with respect to the power series of ε . Then, for the first order,

$$i Z_{\mathbf{k}}^{(1)} T_0 - \omega_{\mathbf{k}} Z_{\mathbf{k}}^{(1)} = 0 \quad (1-28)$$

is obtained. If we take up to the second order of ε , we have

$$\begin{aligned}
 i Z_k^{(2)} T_0 - \omega_k Z_k^{(2)} &= -i Z_k^{(1)} T_1 + \\
 &\sum_{k_1, k_2} H^{(1)}(k, k_1, k_2) Z_{k_1}^{(1)} Z_{k_2}^{(1)} + \\
 &\sum_{k_1, k_2} H^{(2)}(k, k_1, k_2) Z_{k_1}^{(1)} Z_{k_2}^{(1)*} + \\
 &\sum_{k_1, k_2} H^{(3)}(k, k_1, k_2) Z_{k_1}^{(1)*} Z_{k_2}^{(1)*}
 \end{aligned}
 \tag{1-29}$$

Assuming the periodic solution of T_0 , the first order equation is immediately solved as

$$Z_k^{(1)} = X_k^{(1)} \exp(-i \omega_k T_0). \tag{1-30}$$

$X_k^{(1)}$ is an arbitrary function which is independent of T_0 . Substituting (1-30) to the second order of (1-29), we have

$$\begin{aligned}
 i Z_k^{(2)} T_0 - \omega_k Z_k^{(2)} &= -i X_k^{(1)} T_1 \exp(-i \omega_k T_0) + \\
 &\sum_{k_1, k_2} H^{(1)}(k, k_1, k_2) X_{k_1}^{(1)} X_{k_2}^{(1)} \exp\{-i(\omega_{k_1} + \omega_{k_2}) T_0\} + \\
 &\sum_{k_1, k_2} H^{(2)}(k, k_1, k_2) X_{k_1}^{(1)} X_{k_2}^{(1)*} \exp\{-i(\omega_{k_1} - \omega_{k_2}) T_0\} + \\
 &\sum_{k_1, k_2} H^{(3)}(k, k_1, k_2) X_{k_1}^{(1)*} X_{k_2}^{(1)*} \exp\{i(\omega_{k_1} + \omega_{k_2}) T_0\}.
 \end{aligned}
 \tag{1-31}$$

In this equation, we should notice to combinations for the first term of the right-hand side with another term, say the second term, in the right-hand side. These terms are summed up to the following way as

$$\begin{aligned}
 - [i X_k^{(1)} T_1 - \sum_{k_1, k_2} H^{(1)}(k, k_1, k_2) X_{k_1}^{(1)} X_{k_2}^{(1)} \\
 \exp\{-i(\omega_{k_1} + \omega_{k_2} - \omega_k) T_0\}] \exp(-i \omega_k T_0).
 \end{aligned}
 \tag{1-32}$$

If under the summation Σ_k , two conditions

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k} \quad \text{and} \quad \omega_{k_1} + \omega_{k_2} \sim \omega_k \quad (1-33)$$

are simultaneously satisfied, then the time dependence of (1-32) are proportional to $\exp(-i\omega_k T_0)$. If there exists such a term in the equation, the solution $Z_k^{(2)}$ of (1-31) is known to diverge with respect to time T_0 . To avoid the divergence of the solution, we should recognize the whole sum of the terms in [] of (1-32) to be zero. In other words, under the condition (1-33),

$$i X_k^{(1)} T_1 - \sum_{k_1, k_2} H^{(1)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) X_{k_1}^{(1)} X_{k_2}^{(1)} = 0 \quad (1-34)$$

should be satisfied. By virtue of (1-34), $X_k^{(1)}$ is determined with respect to T_1 . The case of another combination is discussed in a similar manner. If the conditions (1-33) are not satisfied simultaneously at all, only the equation

$$i X_k^{(1)} T_1 = 0$$

is required. It means that $X_k^{(1)}$ is independent of T_1 . In reality, as for the deep-water gravity waves, the relations (1-33) are not satisfied (see, for example Kinsman(1965)) so that $X_k^{(1)}$ is constant up to this order. By use of this result, the equation (1-31) is easily solved for $Z_k^{(2)}$.

As the next step, the solution $Z_k^{(2)}$ is substituted in the third order equation and the calculation is executed in the similar manner, then the conditions corresponding to (1-33) are described as

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k} + \mathbf{k}_3 \quad \text{and} \quad \omega_{k_1} + \omega_{k_2} \sim \omega_k + \omega_{k_3} \quad (1-35)$$

These are called the RESONANCE CONDITION of deep-water gravity waves. The condition that the solution is valid for the long time is determined by a similar equation to (1-34) and it represents the T_2 dependence of the first order solution $X_k^{(1)}$. This is known as ZAKHAROV TYPE EQUATION and is discussed in Chapter 3 of this paper. The properties of the equation are precisely interpreted in Appendix III ~ IX.

There could be many other derivations to obtain the mutual interaction equation for water waves. The most formal treatment of the theory by use of CANONICAL THEORY is briefly interpreted in Appendix II. These treatment was applied to the stochastic problems in wind wave field by West(1981) slightly different manner from that discussed in this paper.

CHAPTER 2 EXPERIMENT IN A WAVE BASIN

2. 1 Foreword

In this Chapter, the experiment of non-linear resonant wave interactions performed in the SHIP EXPERIMENT BASIN of the Ship Research Institute (see, Tomita & Sawada(1987)) is described.

Long-time evolution of tertiary resonant waves has not yet been observed in a wave flume. Hence, the experiment is carried out to detect the evolution at the locations spreading widely in a flume. The investigation is performed to find what amount of interaction occurs under several conditions being prescribed.

In this experiment, we choose the simplest feature for examining the resonant interaction phenomena of growing up of the tertiary wave by the perpendicularly intersecting two trains of waves generated with the wave-makers. According to the theory of resonant wave interaction discussed in Chapter 1, the resonance takes place under the condition

$$\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 + \mathbf{k}_4 = 0, \quad \omega_1 - \omega_2 - \omega_3 + \omega_4 \sim 0 \quad (2-1)$$

In particular, in this experiment, $\mathbf{k}_1 = \mathbf{k}_4$, $\omega_1 = \omega_4$ and \mathbf{k}_1 is orthogonal to \mathbf{k}_2 . In this case, (2-1) are solved with respect to $\gamma = \omega_1 / \omega_2$ so that the exact resonance condition is given by $\gamma = 1.736 \dots$. Under this condition, the short time behavior of tertiary wave was discussed by Phillips(1960) and Longuet-Higgins(1962) theoretically, to which we referred briefly in Chapter 1. The experimental studies were also made by Longuet-Higgins & Smith(1966) and McGoldrick, Phillips, Huang & Hodgson(1966) in the smaller wave tanks with the sizes of not exceeding 3 meters square. All these investigations mentioned above were confined to discuss the initial growth of tertiary wave and to verify its growth rate. On the contrary, in our experiment, the observations of long term development of tertiary waves are carried out by use of a comparatively large basin.

Several remarkable results are obtained in this experiment. Above all, it is confirmed that the large amplitude resonant waves which are comparable to that of primary waves appear at the longer fetches than those in previous experiments. These resonant waves travel in the direction which the theory predicts. Moreover, resonant waves are directly observed by photo as an evidence of their existings, for the first time in the field of pure gravity-waves. We examine in the next place the short fetch behavior of resonant wave growth to compare it

with those of the papers above. Finally, we advance further to the long fetch behavior of resonant waves and find the recurrence properties (see for example Waters & Ford(1966)) of interaction among gravity waves. The results are compared with the theory given by Zakharov(1968) which could be applied to the case of this experiment.

In addition to these studies, the observation of the resonant interacting wave system by photographic technique was recently carried out by Strizhkin & Ralentnev(1986) in real open ocean.

2. 2 Description of the Apparatus

As is seen in Fig- 2 - 1, the basin has the size of 80 m in length, 80 m in width and 4.5m in depth. Two wave-makers are installed in the adjacent sides of the basin. The first one is plunger type of 54 m in width driven with 24 sets of 6kW inertia motor, the second one is flap type of 80 m in width driven with two sets of 90kW DC motor. Many trains of waves advancing in different directions can be generated with them. There exists absorbing artificial beaches at the opposite side of each wave-maker. The precise specification of the facility is explained in Shiba(1961) and Takaishi et.al.(1973a, b).

All wave gauges are capacitance type with nominal precision of $\pm 1\%$ and are arranged on the wire rope suspended above the surface of the basin. Each probe is fixed vertically by anchor settled on the bottom. Three examples of the measurement are shown in Fig- 2 - 2. In this figure, the cases that (a) first wave only, (b) second wave only and (c) both waves are simultaneously generated are shown. Upper six rows represent the water surface variations detected by each probe, lower two rows are for the strokes of both wave-makers.

Data collection system is schematically drawn in Fig- 2 - 3. Output signals are sent into the recorder and they are also transferred into disk of a desktop computer through AD converter. Length of each run is limited to 200 seconds for suppressing the effect of wave reflection. Data are digitized every 0.1 second so that we keep the Nyquist frequency as 5Hz. This is sufficiently large value for the present problem.

The effect that the strokes of the wave-makers are finite is considered to be negligible in this experiment. We set the strokes as small as possible to avoid the unfavourable effects of wave breaking, second order wave generation and/or third order wave instability. Nevertheless, diffraction is not completely neglected because the total widths of the partitions are not infinite (diffraction effect was

examined by Ishida et.al.(1980) for this basin applying the wave making theory) . In order to avoid the ambiguity that the height of mechanically generated waves is not constant along its crest, average values for the waves are used.

2. 3 Method of Experiment

Having described the experimental apparatus, let us now turn to the method of measurement. The measurements are executed on two sorts of arrangement of wave gauges shown in Fig- 2 - 4 (Case I) and Fig- 2 - 5 (Case II) . The former is used to reexamine the short term behavior of tertiary wave which was carried out by McGoldrick's experiment and for the first time to detect the direction of propagation of tertiary resonant wave. The latter is used for the measurement of long term development of tertiary resonant waves. At each measurement, the amplitudes of three component waves which would simultaneously exist in the basin are estimated by the power spectral analysis by means of F F T as follows

$$A_k^2 / 2 = (P_{k-1} + P_k + P_{k+1}) \Delta f . \quad (2 - 2)$$

In this equation, A_k and P_k denote the amplitude and component energy density corresponding to the frequency $f = k \Delta f$ ($\Delta f = 0.0098 \text{ sec}^{-1}$) . As is well known in spectral analysis, the energy at single frequency is apt to disperse to its neighbourhoods caused by that the length of data is finite. The precision of this method is tested by aid of dummy data made with electric oscillator. By this test a single component of energy is apparently broadened in width of $\pm 10 \Delta f$ band at the attenuation of -30 d B. Considering the noise property of real data, the band width of $3 \Delta f = 0.029 \text{ sec}^{-1}$ is adopted as shown in (2 - 2) . By using (2 - 2) , restoration ratio of the test data is about 97%.

Our experimental situation and the size of facility lie between most of smaller-scale indoor laboratories and large natural sea field. So the unfavourable affections caused by viscosity and capillarity of water are negligible. All the works are conducted during calm weather, because the basin is in open air. Several runs are tested and checked for inspection over the total inevitable effects due to deformation of waves by wind, reflection, diffraction, breaking, instability of waves and interference with sensors. The primary waves detected repeatedly at the positions closely located as Fig- 2 - 4 show a good agreement in each other. However, the records of the tertiary wave fluctuates with

about 8% of standard deviation. For the frequency, although the motor speeds could be kept constant to within 0.16%, the spectral estimate (2 - 2) has a width of Δf so that the precision of γ is evaluated to $\Delta \gamma = \Delta f / f_2 \sim 0.017$.

The elements of the mechanically generated waves used in the experiment are shown in Table-2-1.

2. 4 Initial Growth of Tertiary Resonant Wave

First of all, we examine whether the tertiary resonant wave k_3 predicted by the theory grows in a basin or not, when we generate a pair of waves k_1 and k_2 mechanically by the wave-makers. An example is shown in Fig-2-6. In this case, $\gamma = 1.793$ and the sensor is located at 45 m from the first wave-maker (nearly mid-point of the basin). In this figure, there appear clearly three line spectra, the lower two lines corresponding to $f_1 = \omega_1 / 2\pi = 1.016$ and $f_2 = \omega_2 / 2\pi = 0.566$ are due to the waves generated by the wave-makers. Remaining one found in higher range is the wave generated by the waves of frequencies f_1 and f_2 . The frequency of this component is $f_3 = 1.475$ and it just agrees with the theoretically predicted $2f_1 - f_2 = 1.466$ within the resolution $\Delta f = 0.0098$. This relation holds good in every case of different values of f_1 and f_2 . From this Figure, one can see that the resonant wave which is to be a third order quantity in theory exceeds the other second order harmonic components and the amount reaches as 50~60% of the first order primary wave. This ratio is more than twice as large as those reported in the previous experiments.

In order for reexamining the previous experimental results, we evaluate the initial growth rate of resonant waves and its dependence upon the frequency ratio γ of the primary waves. As explained in § 1. 3, the initial growth rate G is connected to observable quantities such that

$$A_3 / d (A_1 k_1)^2 (A_2 k_2) = G(\gamma, \theta) | \sin \delta k d / \delta k d |, \quad (2-3-1)$$

$$\frac{2\delta k}{k_3} = - \left(\frac{4}{2\gamma\theta - 1} - \frac{8\gamma\theta^3}{4\gamma\theta^4 + 1} \right) (\gamma - \gamma\theta). \quad (2-3-2)$$

Where, d is the fetch of interaction, δk is the detuning wave-number of primary waves and θ is the angle of intersection.

The initial growth rate G was evaluated 0.442 when $\theta = \pi / 2$ and $\gamma_0 = 1.736$ (the value of G is nearly constant with γ around γ_0). In Fig- 2 - 7, the values of the left-hand side of (2 - 3 - 1) calculated from the measurement data at the location in Fig- 2 - 4 (Case I) are shown against γ . In this case, the wave gauges are located near to the wave-maker 1 to obtain the initial growth data. The solid curve is drawn by the right-hand side of (2 - 3) fitted by inspection with G and γ as parameters. From Fig- 2 - 7, it is estimated that $G = 0.50$ and $\gamma_0 = 1.79$. A comparison with McGoldrick's result is shown in Table- 2 - 2. In this initial stage, the results of γ_0 agree fairly well and are somewhat greater than that of the theory. This fact will be partly explained by the concept of NON-LINEAR RESONANCE CONDITION introduced in Chapter 3. The value G in this experiment lies between the value of their experiment and the classical theory.

Also by means of this location of wave gauges (these six gauges are tightly attached to a stainless steel bar with the mutual distances of 0.45m, 1.05m, 1.20m, 0.60m, 0.30m as consisting a linear array), the determination of the direction of tertiary wave which has not executed in the previous papers is examined. By the theory due to Longuet-Higgins (1962), the angle of tertiary wave to the primary first wave is predicted 9.24 degrees for the case of exact resonance.

Defining the mutual distance between wave gauges D_{12} and the relative angle to the wave α shown as in Fig- 2 - 8, the phase difference ϕ_{12} of the wave for D_{12} is written as

$$\phi_{12} = k D_{12} \sin \alpha \quad , \quad (2 - 4)$$

where, k is the wave-number concerned. Otherwise, phase difference can be calculated from the data obtained at two wave gauges by their CROSS SPECTRUM. If co-spectrum and quadrature-spectrum are expressed as C_{12} and Q_{12} , ϕ_{12} is correlated by them as

$$\phi_{12} = \tan^{-1} (Q_{12} / C_{12}) \quad . \quad (2 - 5)$$

In Fig- 2 - 9, we show the coherence among the data measured with the wave gauges 1 and 3. Although there appears some broadening around the second primary wave, the coherence is almost nearly unity at around the three wave frequencies considered here. Fig- 2 - 10 shows the

phase spectrum of this data. Choosing every pair of gauges from six, the phase $\delta = \phi / \pi$ of three waves f_1, f_2, f_3 is described against D_{1m} (D_{1m} ($l, m=1, 2, \dots, 6$) is distributed not to be equal for every pair of the gauges) in Fig-2-11 (a), (b), (c) respectively. Using the data $k_1=3.993, k_2=1.291, k_3=8.188$ in the formulae

$$\delta_n = (k_n D_{1,m} / \pi) \sin \alpha_n, \quad (2-6)$$

where $n = 1, 2, 3$ and $l, m = 1, 2, \dots, 6,$

we can determine $\alpha_1, \alpha_2, \alpha_3$ from the tangent of each plot. The straight lines in Fig-2-11 are obtained by means of the least square method. By these Figures, we can estimate that $\alpha_1=1.09, \alpha_2=73.40$ and $\alpha_3=-7.85$ degrees so that the direction of tertiary wave from the first primary wave is $\alpha_3 - \alpha_1 = -8.94$ degrees, while the theoretical prediction in this case is -9.19 degrees. We can recognize that the agreement of both values is satisfactory.

2. 5 Long Term Evolution of Tertiary Resonant Wave

In this section, we investigate the long term behavior of the tertiary resonant wave. In order to perform this task, wave gauges are arranged as shown in Fig-2-5 (Case II). Six wave gauges are set at the distance from the first wave-maker of 26.56 m, 35.96 m, 41.15 m, 45.36 m, 50.95 m and 61.15 m along the direction of tertiary waves. They are the very longer fetches than those of the Case I and those of the previous works (their maximum span of observation is about 15 m after transforming the size to the present experiments).

The results are explained in the following. In Fig-2-12 through Fig-2-18, the amplitudes of tertiary waves are plotted against the distances along the direction of propagation.

For describing the observational results clearly, we explain them corresponding to the experimental conditions in order:

- 1) $\gamma \sim 1.72$ (nearly resonant), A_1 and A_2 (~ 2.5 cm) are both small. {Fig-2-12}

In this case, the growth of resonant waves are nearly straight. The broken line shows the theory of Longuet-Higgins (1962) (equation (2-3)). The long fetch behavior can be explained in this case qualitatively by the classical theory.

2) $\gamma \sim 1.72$ (nearly resonant), A_1 ($\sim 4\text{cm}$) is larger than the case ①.
{Fig- 2 - 1 3 }

While A_2 ($\sim 2.5\text{cm}$) is as same as the case ①, resonance does not strongly occur and the amplitude of tertiary wave is in every point small. The curve represents the quasi-stationary solution given by (3 - 8) by means of the Zakharov theory.

3) $\gamma \sim 1.79$ (off resonant), A_1 is small and A_2 ($\sim 5\text{cm}$) is moderate.
{Fig- 2 - 1 4 }

In this case, A_3 is nearly constant (slowly varying) throughout the fetch where the measurements are made. The manner of variations looks almost parallel and the values are found larger as A_1 increases from 1.80 to 2.84. In the last case (A_1 is the largest), the values of A_3 amounts to about 1.5 cm. The appreciable values of resonant waves are observed in the first time in such a off resonant cases.

4) $\gamma \sim 1.79$ (off resonant), A_1 is larger than the case 3) while A_2 is small. {Fig- 2 - 1 5 }

This is rather curious result. Although the condition is so far from the case 1), the growth of A_3 is clearly straight. The broken line in this figure is the theoretical one like the item 1) (omitting the de-tuning factor). The dashed-and-dotted line is determined by the least square fitting. Looking at the discrepancy between both lines, it suggests that in this case, a sort of non-linear resonance condition including the amplitude dependence to the wave velocity would hold and it suppresses the free evolution of tertiary wave.

5) $\gamma \sim 1.79$ (off resonant), A_1 is larger than case 3). {Fig- 2 - 1 6 }
 A_3 clearly decreases as the fetch increases and diminish to zero (recurrence phenomena) instead that the asymptotic steady states take place in a longer fetch.

6) $\gamma \sim 1.82$ is larger, A_1 and A_2 are both large. {Fig- 2 - 1 7 }

In this case, it is characteristic under this condition that the magnitudes of A_3 decrease initially as the fetch increases and then grow up once again. Subsequently resonant waves repeat the same process. However this is not sure in the present experiment because the length of the basin is not enough long to pursuit this character. This tendency appears the faster (at the shorter fetch) with the larger A_1 .

7) The largest wave obtained in this experiment is shown in Fig-2-18. In this experiment, tertiary resonant waves has never exceeded 2.5 cm in amplitude (5 cm in wave height). This limitation may depend upon the wave-steepness of the primary waves used in this experiment. Local breaking of waves is apt to arise particularly in such a composed wave system that mechanically and spontaneously generated waves consist of a comparatively wide spread frequency components. These local breakers possibly prevent the resonance mechanisms from being sufficiently enhanced.

8) In the case of γ far from γ_0 , say $\gamma < 1.6$ or $\gamma > 2.0$, it is verified that no wave is generated at all.

In general, the straight resonant growth is seriously dependent on the conditions among the frequencies and amplitudes of primary waves. On the contrary, the recursive resonant growth occurs in somewhat soft conditions whereas the maximum values of them are comparative to the former. The decreasing of amplitudes of tertiary waves at the longer fetch rather reveals that the strong interaction takes place even in this region, otherwise the resonant waves which are once generated at shorter fetch would travel to the outer region without decaying their amplitude at all.

The tertiary resonant waves generated by mutual interaction of primary waves can be observed by the naked eye in this experiment. Since the wave velocity of tertiary wave is much less than the primary waves, it can be left in the basin after stopping the wave-makers and passing the primary waves away to the absorbing beaches. This fact is another confirmation that these tertiary waves are free waves in accordance with the theory. Three photographs on the experiment of the generated resonant wave are shown in Fig-2-19. The direct photographic observation of deep-water gravity wave interaction had not been known in the past. From the picture of Fig-2-19 (c), wavelength of the tertiary wave taken in the photo is measured as 72.5 cm. While the theoretical length is 71.7 cm.

CHAPTER 3 NUMERICAL SOLUTION OF ZAKHAROV EQUATION

3. 1 Foreword

The non-linear theory described in § 1. 5 gives an integro-differential equation which governs the slow variations of first order amplitude and phase components among multiple directional waves. This type of equation was first derived by Zakharov(1968), and is called the ZAKHAROV EQUATION. In general, it is difficult even to obtain the solution of this equation by numerical method, not to mention to solve it analytically. So, the Zakharov equation has never been applied except for the stability problems of monochromatic wave train.

In this Chapter, we deal with this equation in the most important case of three waves mutual interaction by regarding it as a system of ordinary differential equations. At first, a simple approximate solution to this system of equations is derived analytically assuming that the energy transfer among waves is not so large. This solution lends itself to consider the resonance condition with the amplitude effect taking into account. In the next place, the measurements at shorter fetches given by McGoldrick et. al. (1966) is successfully compared with this theory. A simple and clear evaluation of the limiting wave height of resonant waves is also put forward in terms of the first primary wave amplitude. The result is confirmed numerically by the repeated execution of long-time numerical integration of this system of equations. Through this calculation, recurrence properties which are found and described to some extent in Chapter 2 are reproduced.

The comparison of the results are made with experiments described in Chapter 2, and the comprehensive discussion on the resonant interaction phenomena are yielded in Chapter 4. At the last section of this Chapter, a related problem on instability properties of a quasi-monochromatic wave train are treated by the same method. The relation of this equation with Hasselmann's energy flux equation among continuous spectral component is interpreted in Appendix III. The relation with Non-linear Schroedinger equation is also explained in Appendix IV.

3. 2 Numerical Experiment

The fundamental integro-differential equation has the form

$$i \frac{\partial B(\mathbf{k}, t)}{\partial t} = \iiint_{-\infty}^{\infty} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \\ B^*(\mathbf{k}_1, t) B(\mathbf{k}_2, t) B(\mathbf{k}_3, t) \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \\ \exp\{i(\omega + \omega_1 - \omega_2 - \omega_3)t\}. \quad (3-1)$$

This is conceptually equivalent to (1-34). In this expression, the symbol δ is Dirac delta-function. The explicit form of the kernel T is presented in Appendix V. Using the quantity B , surface elevation η is expressed as

$$\eta(x, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} (\mathbf{k}/2\omega)^{-1} d\mathbf{k} B(\mathbf{k}, t) \exp(i\mathbf{k} \cdot \mathbf{x} - \omega t). \quad (3-2)$$

Pulling out from (3-1) the three components discussed in Chapter 2, it is transformed into ordinary differential equations as

$$i \frac{dB_1}{dt} = [T_{1111} B_1 B_1^* + \tilde{T}_{1221} B_2 B_2^* + \tilde{T}_{1331} B_3 B_3^*] B_1 + \\ \tilde{T}_{1123} e^{i\Delta\omega_{1123}t} B_1^* B_2 B_3, \quad (3-3-1)$$

$$i \frac{dB_2}{dt} = [\tilde{T}_{2112} B_1 B_1^* + T_{2222} B_2 B_2^* + \tilde{T}_{2332} B_3 B_3^*] B_2 + \\ T_{2311} e^{i\Delta\omega_{2311}t} B_3^* B_1 B_1 \quad (3-3-2)$$

and

$$i \frac{dB_3}{dt} = [\tilde{T}_{3113} B_1 B_1^* + \tilde{T}_{3223} B_2 B_2^* + T_{3333} B_3 B_3^*] B_3 + \\ T_{3211} e^{i\Delta\omega_{3211}t} B_2^* B_1 B_1. \quad (3-3-3)$$

These are actually the six degree non-linear equations with respect to the real and imaginary parts of B .

Where, T_{1234} denotes $T(k_1, k_2, k_3, k_4)$ and conventional notations $\widetilde{T}_{1234} = T_{1234} + T_{1243}$, $\Delta\omega_{1234} = \omega_1 + \omega_2 - \omega_3 - \omega_4$ are used. It is confirmed that this discretized approximation is self-consistent and the other components play no role at least in the first order, if they does not exist a priori. The first terms in the right-hand sides of (3-3-1) ~ (3-3-3) represent the phase velocity effect in tertiary wave interaction which is briefly interpreted in Appendix VI.

Before solving (3-3-1) ~ (3-3-3), we discuss about the conservation laws of this system.

Taking notice on the magnitude of B , the symmetrical property of the equations leads that

$$2 \widetilde{T}_{1123}^{-1} |B_1|^2 + T_{2311}^{-1} |B_2|^2 + T_{3211}^{-1} |B_3|^2 = \text{const}$$

and (3-4-1)

$$T_{2311}^{-1} |B_2|^2 - T_{3211}^{-1} |B_3|^2 = \text{const} . \quad (3-4-2)$$

From the expressions (3-3-1) ~ (3-4-2), one can immediately notice for the energy transfer among these three waves that the first primary wave B_1 shears its energy to B_2 and B_3 for growing them, that is, the energy flows from B_1 toward B_2 and B_3 , or vice versa. The first primary wave B_1 plays the most fundamental role in this interaction and unlike it, the role of the second primary wave B_2 is subsidiary.

Considering that the complex amplitude B has a relation with the actual wave amplitude A as

$$|B(k)| = \pi \left(\frac{2\omega}{k} \right)^{\frac{1}{2}} A(k), \quad (3-5)$$

it leads to

$$|B(k)|^2 = \pi^2 \left(\frac{2g}{\omega} \right) A(k)^2 . \quad (3-6)$$

Because A^2 is proportional to the energy of waves, $|B(k)|^2$ means the wave action (see Leibovich et.al. (1974) or Phillips (1977)) in this system. Conservation laws are interpreted in more details in Appendix VII.

In the next step, we examine an approximate analytical solution

of equation (3-3-1) \sim (3-3-3). In this approximation, we assume that the amplitude of resonant wave is much less than those of the primary waves. By use of this assumption, we neglect the terms containing B_3 in (3-3-1) \sim (3-3-3). In this manner, the amplitudes of the primary waves are regarded as constants so that the quantities in [] of (3-3-1) \sim (3-3-3) should be also constants. They are denoted by θ_1 , θ_2 and θ_3 ($\Delta\omega=0$ is set without loss of generality), that is,

$$\theta_1 = [T_{1111} B_1 B_1^* + \tilde{T}_{1221} B_2 B_2^*],$$

and

$$\theta_2 = [\tilde{T}_{2112} B_1 B_1^* + T_{2222} B_2 B_2^*]$$

$$\theta_3 = [\tilde{T}_{3113} B_1 B_1^* + \tilde{T}_{3223} B_2 B_2^*].$$

Representing $B_n(t) = b_n(t) \exp i \chi_n(t)$, ($n=1,2,3$) under the constraint of b_n , χ_n being real functions, we get from (3-3-1) and (3-3-2) that $b_1(t) = b_{10}$, $b_2(t) = b_{20}$, $\chi_1(t) = -\theta_1 t$ and $\chi_2(t) = -\theta_2 t + \pi/2$. Using them to the last equation (3-3-3), it reduces to

$$\frac{d b_3}{d t} = T_{3211} b_{10}^2 b_{20} \cos \{ (2\theta_1 - \theta_2) t + \chi_3 \}$$

and

$$(3-7-1)$$

$$\frac{d \chi_3}{d t} = -\theta_3 - T_{3211} b_{10}^2 b_{20} b_3^{-1} \sin \{ (2\theta_1 - \theta_2) t + \chi_3 \}$$

$$(3-7-2)$$

which are non-linear equations with respect to b_3 and χ_3 . Considering the initial condotion $b_3=0$, $\chi_3=0$, at $t=0$, we introduce an undetermined constant β as $\chi_3 = -\beta t$ and integrate (3-7-1). The result is

$$b_3 = \{ K / (2\theta_1 - \theta_2 - \beta) \} \sin (2\theta_1 - \theta_2 - \beta) t$$

where $K = T_{3211} b_{10}^2 b_{20}$. Substituting χ_3 and b_3 into (3-7-2), we can determine β as follows

$$-\beta = -\theta_3 - (2\theta_1 - \theta_2 - \beta), \text{ that is, } \beta = \theta_1 - \frac{1}{2}\theta_2 + \frac{1}{2}\theta_3.$$