Thus,

$$\chi_3 = -(\theta_1 - \tfrac{1}{2}\theta_2 + \tfrac{1}{2}\theta_3)\ t\ .$$

Accordingly, time variation of b_3 can be decided as

$$b_3 = \{K / (\theta_1 - \tfrac{1}{2}\theta_2 - \tfrac{1}{2}\theta_3)\}\ \sin(\theta_1 - \tfrac{1}{2}\theta_2 - \tfrac{1}{2}\theta_3)\ t \quad . \tag{3-8}$$

It is easy to verify that this pair of solutions χ_3 and b_3 satisfies the equation (3 − 7 − 1) and (3 − 7 − 2) exactly. In the initial stage of evolution, the solution (3 − 8) reduces to $b_3 = K\,t$, which would be equivqlent to the classical result (Longuet-Higgins(1962)) . In order to verify whether the theory of Zakharov equation to be applicable to the phenomena or not, we now compare (3 − 8) with the experimental results given by McGoldrick et. al. (1966)as the initial growth of tertiary waves. In accordance with their experimental parameters, we rewrite (3 − 8) as

$$a_3 = (4\pi^2 T_{3211})\ (\frac{\omega_3}{\omega_1})\ (\frac{\omega_3}{\omega_2})^{1/2} a_1^2 a_2 d\ , \tag{3-9}$$

where d is the fetch of interaction. We adopt concrete values on the basis of their experiment as $a_1 = 0.32$ cm, $a_2 = 0.895$ cm, $\omega_1 = 16.87$ $\sec^{-1}$, $\omega_2 = 9.65$ $\sec^{-1}$ and $\omega_3 = 24.0$ $\sec^{-1}$. Thus, we can calculate the amplitude of tertiary waves against fetch d by evaluating the coupling coefficient $T_{3211} = 40.964$ from the Zakharov theory. The results under the condition mentioned above, together with the case that $a_2 = 0.45$ cm (one-half of the former) with the symbols ○ and △ respectively are drawn in Fig− 3 − 1 . Their data on two series of experiment show good agreement with the Zakharov theory. In this paper, we call (3 − 8) the QUASI-STATIONARY solution of the equations (3 − 3 − 1) ∼ (3 − 3 − 3) .

Using the quasi-stationary solution verified to be valid immediately before, we consider the NON-LINEAR RESONANCE CONDITION, that is, the dependence of $\gamma_\varnothing$ upon the amplitudes of primary waves. Slight extension to the solution (3 − 8) when $\Delta\omega \neq 0$ yields the modification of its argument as $\theta_1 - \frac{1}{2}\theta_2 - \frac{1}{2}\theta_3 + \frac{1}{2}\Delta\omega$. Therefore, by this approximation the non-linear resonance condition is expressed as

$$\Delta\omega + 2\delta = 0, \qquad (3-10)$$

where δ is given such that

$$2\delta = [2T_{1111} - \widetilde{T}_{2112} - \widetilde{T}_{3113}]\ B_1 B_1^{*} + [2\widetilde{T}_{1221} - T_{2222} - \widetilde{T}_{3223}]\ B_2 B_2^{*}. \qquad (3-11)$$

It is obvious that for the linear resonance condition, (3 − 1 0) merely reduces to $\Delta\omega = 0$ and $\gamma_0 = 1.7357\cdots\cdots$. For evaluating the small correction γ', we assume that the non-linear resonance condition γ_m is expressed by $\gamma_m = \gamma_0 + \gamma'$ and approximate (3 − 1 0) up to the first order of γ'. The result is

$$\gamma' = \frac{-(8\gamma_0^3 - 12\gamma_0^2 + 6\gamma_0 - 1)}{2(6\gamma_0^3 - 12\gamma_0^2 + 6\gamma_0 - 1)}\ (2\delta). \qquad (3-12)$$

(3 − 1 2) together with (3 − 1 1) represents a correction of the resonance condition by finite amplitudes of primary waves.

If we apply the Zakharov's coefficients, the non-dimensional formula is derived that

$$\gamma' = 1.66055\,(a_1 k_1)^2 - 2.74992\,(a_2 k_2)^2. \qquad (3-13)$$

An example for the case calculated in § 3 . 3, that $a_1 = 4.7$ cm, $a_2 = 5$ cm, $\lambda_1 = 1.66$ m and $\lambda_2 = 4.99$ m leads to $\gamma' = 0.051838\cdots\cdots$. Thus we obtain the value $\gamma_m = \gamma_0 + \gamma' = 1.788$ which agrees fairly well with the value $\gamma = 1.800$ adopted in the caluculation.

In the following section, we mention how tertiary wave amplitude A_3 is correlated with the changes of the amplitudes A_1, A_2 and the frequencies ω_1, ω_2 of the primary waves. The details of the numerical procedur are referred to Tomita(1987).

3 . 3 Behavior of the Tertiary Resonant Waves

We can integrate the equation (3 − 3 − 1) ~ (3 − 3 − 3) numerically under the condition that amplitudes A_1 and A_2 are given as concrete values in the experiment. The amplitude A_3 is assumed to be zero initially. The phase difference between the primary waves has no influence upon the results. Corresponding to various values of A_1 and A_2, long time variations of three waves are shown in Fig-3-2, and Fig-3-3. It is shown that the energy exchange occurs among the three waves and the amplitudes of waves vary periodically (not always sinusoidal) and never reach any equillibrium (this problem is discussed in more detail by the analytical investigation of these equations at the latter part of this paper). Fig-3-2 corresponds to the case $\gamma=1.735$(near resonant). Initial values of A_1 are prescribed (a) 1cm (b) 2cm (c) 3cm and (d) 4cm in order, while A_2 is fixed as 5cm. The growth of A_3 is apparently limited. The straight line A_3' in Fig-3-2 (b) is the solution given by Longuet-Higgins(1962). It means that the solution of Zakharov equation reduces close to the classical one in the initial stage $t \ll 1$ as mentioned at the previous section.

On the contrary, when $\gamma=1.800$(off resonant) A_3 grows to some extent according to the increase of A_1 (see Fig-3-3 (a) ~ (e)). Initial values of A_1 are prescribed (a) 2cm (b) 3cm (c) 4cm (d) 4.6cm and (e) 5cm, while A_2 is fixed as 5cm. In Fig-3-3 (d), the amplitude A_3 temporarily exceeds the first order quantity A_1. If we set that $A_1=5$cm initially, the growth of A_3 rather reduces.

In the second place, drawing our attention to the nature that A_3 reaches their maximum values in finite durations in any cases, we investigate the values of the maximum A_{3max} against A_1 with γ as a parameter. A result when A_2 is fixed as 5cm, is shown in Fig-3-4. In Fig-3-4, the parameter R which is square of γ (the exact resonance ratio $\gamma=1.736$ discussed in Chapter 2 corresponds to $R=3.014$) is used. The value of γ is also shown in Fig-3-4. When $R>3.0$, each solution A_{3max} corresponding to different values of R has sharp peak A^M_{3max} in the vicinity of each value A_{1R} without regard to R. The fact is also noticed that in the case $R>4.0$, each solution A_{3max} as a function of A_1, is nearly identical without respect to R. It is obvious from the mathematical point of view. The reason is understood that the Zakharov coefficients T_{abcd} (a, b, c, d=1, 2, 3) does not vary so much with R. Whereas, physically speaking, it is not so obvious. We merely point out that A^M_{3max} exceeds the highest limit of gravity wave, hence the formulation of the theory up to the third order of wave steepness would be insufficient under this condition.

A^M_{3max} is a quantity which is characteristic to express the intensity of resonant wave interactions. Using the results discussed in Appendix VII, we have a criterion about the limit of maximum amplitude of tertiary wave A^M_{3max} that it depends only on A_1 (not on A_2) linearly such that

$$A^{M}_{3max}=0.844A_1 \quad . \qquad (3-14)$$

The maximum value A_{3max} of tertiary resonant waves could grow to the extent of 84% of the first primary wave which generates it. In the case $R<2.9$, the curves run close to the abscissa, that is, the small part of energy can be transfered. For the case of amplitude of the second wave A_2 is 10 cm, the maximum of tertiary wave A_{3max} is shown in Fig－3－5. The broken line in Fig－3－5 is (3－14) which passes through the each maximum of A_{3max}, in this paper, we express it as A^{M}_{3max}. We could not examine this formula (3－14) directly, because it is difficult to generate sufficiently large amplitude wave which has non-deformed, non-breaking crest lines of constant height with the wave-makers used in this experiment.

Finally, we execute several numerical integration by arranging the initial values of the amplitudes of two primary waves as real value recorded in the experiment. Examples are displayed in Fig－3－6～Fig－3－7. As is explained in Chapter 2, two primary waves could not directly be compared with the theoretical ones because their amplitude are affected more intensely by inevitable effect of wave diffraction than that of interactions. The plotting is done only for tertiary waves. In the theories, phenomena are assumed to be uniform in space and vary with time. On the contrary, for the experiment, we make up the stationary state in a basin and detect the spatial variation of wave amplitudes with several wave gauges. By this reason, the wave amplitude of tertiary waves are drawn against spatial fetch d. Fig－3－6 for $R=2.97$ ($\gamma=1.72$), Fig－3－7 for $R=3.21$ ($\gamma=1.79$) show the data with the Zakharov's theoretical values. Between the both examples, the manner of variation of A_3 are somewhat different, nevertheless the agreement of the data with the theory are fairy well. However, as is seen at the last example, Fig－3－7 (c), experimental data do not amount so large as half as that of the Zakharov theory when the wave height is extremely large.

In order to show the applicability of the theoretical results to the scale of actual sea, we present a summary concerning the similarity problem in Appendix Ⅷ. Analytical properties of interaction equations (3－3－1)～(3－3－3) are briefly investigated in Appendix Ⅸ with the proof for existing of the steady state asymptotic solution corresponding to the specific amplitudes of first and second primary waves.

3.4 Instability Properties of a Wave Train

We must mention first of all, the famous study by Benjamin & Feir (1967) with regard to this problem. By their theory, a finite amplitude deep-water wave train is unstable to the small subharmonic disturbances whose components have a pair of side-bands of ω, say $\omega+\Delta\omega$ and $\omega-\Delta\omega$. They restricted themselves to one-dimensional problem that disturbing waves advance in the same direction as the primary wave. Recently, Crawford, Lake, Saffman & Yuen(1981) by means of the Zakharov equation, MacLean(1982)by use of the exact Eulerian equations calculated the domain of stability to two-dimensional perturbations in the framework of linear instability theory. Su, Bergin, Marler & Myrick(1982), Su (1982) also carried out the experimental studies in a very long wave flume in open air and indicated the importance of the two-dimensional perturbations to the stability of steep gravity waves. Observations on a modulational characteristics of wind waves were conducted by Mase et.al. (1985) and Donelan(1987)in actual sea. The former authors successfully compared their observational data with the computational results from the Zakharov equation. We should also refer the studies on a instability of non-linear standing water waves elaborated by Okamura(1984, 1985) using the Zakharov equation. In this section, we utilize (3−3−1) ~ (3−3−3) as they are, to investigate a monochromatic wave which is exerted by two-dimensional small perturbations.

To this type of problems, B_1 is recognized as a primary wave and B_2, B_3 as a pair of side-band perturbations advancing in the different directions. In Fig−3−8 one can see an example of the long time behavior of each components A_1, A_2 and A_3. The perturbation components rise up spike-wise intermittently in all the cases to be examined. In these calculations, the magnitudes of small perturbations A_2 and A_3 are initially set 10^{-6} times as large as that of primary wave A_1. The height and the recurring period of spikes are intrinsically dependent on the wavelengths and directions of two perturbational component waves. Thus we examine the waves of wave-numbers $\mathbf{k}_1=K_1(1,\ 0)$, $\mathbf{k}_2=K_1(1+p,\ q)$ and $\mathbf{k}_3=K_1(1-p,\ -q)$ in the regions $0<p<1.2$, $0<q<0.5$. The vector $K_1(p,\ q)$ is taken as a modulational wave-number. The instability criterion is defined that the side-band amplitudes exceeds 10^{-4} times as large as that of the primary wave. The domain of instability calculated by the Zakharov equation are shown in Fig−3−9. Small circles mean the unstable couples $(p,\ q)$. The stable results are not illustrated in the Figure, e.g., the wave is stable in the region except where is filled by the grid of the small circles.

Solid curves drawn in the same Figures represent the boundary of the domains of instability by means of linear stability theory after MacLean (1982). From the numerical experiment executed in this time, side-band components rise up abruptly at the outer side boundary of the wave-number space.

The stability property of a wave train can be investigated in the same manner as resonant problem, conversely, the stability property has not been sufficiently taken into accounts in studying the resonant interactions. In this study, Zakharov equation was discretized into the most important three wave components. However, from the stand point mentioned in this section, the affection on the resonant interaction properties by other components must be investigated. There would be certain contributions through the instability and phase speed effect exerted on primary waves by other components neglected in this paper. To a further step, the computation with a great many components are desirable for directly simulating the actual ocean wave spectra, possibly by use of super computer. It will be an issue to be treated more comprehensively in the future studies.

CHAPTER 4 CHARACTERISTICS OF THE RESONANT WAVE INTERACTION

4. 1 Outline of the Preceding Chapters

In the previous chapter, we explained the classical theory due to Longuet-Higgins(1962)in§ 1. 3 and the more comprehensive theoretical approach to long term evolution of resonant interactions in §§ 1. 4 and 1. 5. The experimental results shown in Chapter 2 revealed that the classical theory is insufficient to describe the observational results. In Chapter 3, the Zakharov equation which is applicable to long term variation of non-linear waves was numerically integrated and the experimental data were partly confirmed to agree with the theory in several examples. In this Chapter, the experimental data are compared with these theories in more entire point of view. For this purpose, we summarize the facts obtained in the preceding Chapters as follows:

1) The existence of tertiary wave generated by resonant interaction is verified experimentally. The tertiary wave which grows up to 62% as large as the first primary wave is detected when $\gamma = 1.79$ and $d = 45.36$m.

2) In short fetches, the growth rate of resonant waves is somewhat smaller than that measured by McGoldrick et.al.(1966) and greater than theoretical value given by Longuet-Higgins(1962) by 18%. Tertiary wave growth takes place most strongly at the value of $\gamma = 1.78$ which is slightly different from the exact resonance condition $\gamma_0 = 1.736$. This is partially interpreted by the non-linear correction of the resonance condition.

3) The measurements done by McGoldrick et.al. in the short fetch are completely accounted by the Zakharov theory.

4) The direction of propagation of generated tertiary resonant waves are determined about 9 degrees which is identical with the theory.

5) In longer fetches, tertiary wave grows up to their maximums and diminishes its amplitude. The fetch lengths for the recurrences depend upon the amplitudes of primary waves A_1 and A_2. This fact can not be explained by classical theory (expressed in (1 − 9)).

6) In general, the behavior of tertiary waves is affected not only by the frequency ratio γ which is related to resonant condition, but also by

the amplitudes of primary waves A_1 and A_2.

7) By comparison of the Zakharov equation with the observational data, we can see that the theory explains the evolution of tertiary waves at the case of small steepness of each waves. However, the discrepancies become large with increase of the wave steepness.

8) An approximate analytical solution of Zakharov equation (3 − 8) is proposed. The observational results are explained by this solution when energy transfer among waves is comparatively weak.

9) By solving the Zakharov equation repeatedly, the maximum values A_{3max} realized by tertiary waves are determined against A_1 with γ as a parameter.

As the quantitiy A_{3max} is suitable to discuss about the entire characteristics of resonant wave interactions, we rearrange the data to be compared with theories through this concept.

4. 2 Comparison with Classical Theory

The recurrence properties of tertiary waves are explained even by the theory of Longuet-Higgins(1962) if we recognize them as an effect of detuning of the frequency ratio γ to its prescribed value γ_0. According to this theory, the maximum value A_{3max} to be realized is yielded by (1 − 14). In Fig−4 − 1, we take the theoretical values to the abscissa and those of experimental values as the ordinate and plot the points in the graph of dispersion. If theory and experiment agree with each other, the points should be distributed on a line drawn in Fig− 4 − 1. The result scatters to a large extent. This means that the theory can not explain the experimental results. Taking into consideration that there are results for many cases of γ in Fig− 4 − 1, we classify them into three main categories. The symbol △ corresponds to the case $\gamma \sim 1.72$(nearly resonant case). In this case, all the data run close to the x-axis. It means that resonant waves do not so evolved as the classical theory predicts. Symbols □ correspond the case $\gamma \sim 1.79$ (not so close to the resonant case). The data are seen to wind themselves around the line and comparatively near to it. The dispersion seems not to be random. The points are distributed higher for small A_{3max} and lower for large A_{3max} than the solid line. The case $\gamma > 1.82$ is plotted by the symbol ○. In this case, all the points are plotted above the line, in other words, the measured values are always larger than that of the theory. The reasons why the theory and experiment are not identical in general is suggested as follows:

Firstly, the velocity of tertiary wave changes by the influence of non-linear amplitude dispersion. Velocity becomes slightly larger with increase of the wave amplitude. As a result, resonance system of wave-wave interactions turns out of tune. On the contrary, for the case that resonance condition is not so closely satisfied, exact resonance can be preserved by a slight-detuning to compensate for amplitude dispersion inferred by Phillips(1977). Moreover, primary waves shed their energy to the other waves to intensify the growth of resonant waves. These effects were not considered in this classical theory.

4.3 Comparison with Zakharov's Theory

In order to clarify the effect of the primary wave amplitude A_1 to the growth of tertiary resonant wave quantitatively, we examine the dependence of the maximum amplitude of tertiary wave A_{3max} on A_1 in the sequel. The experimental values of A_{3max} are plotted this time against A_1 in Fig-4-2. There seems no clear tendency in Fig-4-2. We arrange these data in the following manner. As same as the prvious section, the set of data is classified by the magnitudes of γ.

The case $\gamma \sim 1.72$ is shown in Fig-4-3. Theoretical curve calculated by means of Zakharov equation is also shown in Fig-4-3. Taking various noise described in Chapter 2 into considerations, the agreement of the theoretical prediction with the acquired data is fairy well in this case. Fig-4-4 shows for the value $\gamma \sim 1.79$. It is very characteristic in this figure that the theoretical curve expresses the existence of strong resonance in the vicinity of $A_1 \sim 4.0$ cm. The measured data agree well with this characteristics. Although the sharp peak for A_{3max} is not observed experimentally, the discrepancy might be caused by that the waves made with wave-makers are not perfectly monochromatic, so the critical condition demanded by the theory for the peaks would not be realized. On the other hand, higher order effects which are not considered in the theory might have an influence under such a subtle condition. In Fig-4-5, the case $\gamma > 1.82$ is totally plotted. The data are somewhat widely scattered in this Figure, however considering the instability property of waves at the large amplitude, it is concluded that the entire behaviors obtained in the experiment could be explained by the theory of Zakharov equation.

4. 4 Discussion

It is confirmed that the long term evolution of tertiary resonant waves are not explained by the classical theory. On the other hand, by the comparisons of the experiment with the theory of Zakharov we can conclude that this theory is applicable to this sort of phenomena. It could predict the evolution of tertiary resonant waves under the conditions that the wave steepness $H/L<0.05$ (the reproducible experiment was conducted to the wave whose steepness is less than 0.05) and the frequency ratio $1.58<\gamma<1.90$. It is the point left as an open question when one applies this sort of equations derived by the singular perturbation method. Although these criteria are not determined directly by the experiment, they are discussed briefly in AppendixⅦ.

After all we summarize the over all properties of the generation of tertiary resonant wave by perpendicularly intersecting two primary waves as follows:

1) In general, resonant waves show a spatial (temporal) recurrence (periodicity). The non-linear resonant wave interaction phenomena are interpreted by a third order slowly varying theory using the Zakharov equation.

2) Growth rate G for short term development is proportional to the square of the first primary wave amplitude A_1. Classical theory is valid only for this region.

3) The resonance takes place most strongly at the "off-resonance" condition $\gamma\sim1.79$ in terms of the linear dispersion relation. Introduction of the concept "non-linear resonance" is necessary.

4) To the values of γ less than γ_0, there exists no strong resonance and A_{3max} approaches to a small constant value without respect to A_1.

5) For the cases $\gamma<1.6$ and $\gamma>2.2$, tertiary wave does not appear at all in the experiment.

There were several reports including Snodgrass et.al.(1966) who pointed out the importance of wave-wave interactions in a seaway. Mollo-Christensen & Ramamonjiarisoa(1978, 1982) proposed a new model for ocean waves described by the presence of wave groups in a random wave field. Chereskin & Mollo-Christensen(1985) conducted an experimental study about the amplitude and phase modulation of a one-dimensional wave flume. The coherency of narrow-band weakly non-linear one-dimensional wave system is pointed out by their papers. If such a coherent property is predominant in the ocean waves, resonant interaction would take place more intensely than considered in a model of random wave field. Recently, Sand(1988)reported the topics in the problems of wave forces as a environmental conditions to ocean structures. In the field of research concerning the mooring of off-shore floating structures, for example, investigations into non-linear properties of sea waves will play an essential role in the near future.

The present author(1988b, c) also investigated the wave group characteristics by use of the data obtained with wave buoy at the North Pacific Ocean during 1983～1984. It might be a manifestation of non-linear modulation property of wind waves indicated by Mase et.al.(1985) or Li et.al.(1987). In the present time, observational wave data are not enough to make clear the mechanisms of this sort of unsteady non-linear processes in sea waves. Application of more exhaustive analysis techniques such as INSTANTANEOUS SPECTRUM proposed by Bendat & Piersol(1967) and/or INVERSE SCATTERING METHOD founded by Zakharov & Shabat(1972) and interpreted by Sobey & Colman(1982, 1983) in the context of sea waves seem to be necessary.

Although it is a future problem that the investigations are executed for the more general cases, four wave mutual interactions etc., the co-operative method of study within theory, calculation, experiment and obserbation is indispensable for such a non-linear problem.

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Appendix I Coefficients $H^{(N)}$, $F^{(N)}$

Coefficients in (1 − 24) are yielded as follows;

$$H^{(1)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = 1/(2\sqrt{2})\,[(gk_2/kk_1)^{1/4}D^{(1)}(\mathbf{k}_1, \mathbf{k}_2) + (gk/k_1k_2)^{1/4}D^{(2)}(\mathbf{k}_1, \mathbf{k}_2) - (gkk_1k_2)^{1/4}D^{(3)}(\mathbf{k}_1, \mathbf{k}_2)]\,\delta_{0-1-2}, \quad (I-1)$$

$$H^{(2)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = 1/(2\sqrt{2})\,[(gk_2/kk_1)^{1/4}D^{(1)}(\mathbf{k}_1, -\mathbf{k}_2) - (gk_1/kk_2)^{1/4}D^{(1)}(-\mathbf{k}^2, \mathbf{k}^1) - (gk/k_1k_2)^{1/4}\{D^{(2)}(-\mathbf{k}^2, \mathbf{k}^1) + D^{(2)}(\mathbf{k}_1, -\mathbf{k}_2)\} - (gkk_1k_2)^{1/4}\{D^{(3)}(\mathbf{k}_1, -\mathbf{k}_2) + D^{(3)}(-\mathbf{k}^2, \mathbf{k}^1)\}]\,\delta_{0+1-2}, \quad (I-2)$$

$$H^{(3)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = 1/(2\sqrt{2})\,[(gk_2/kk_1)^{1/4}D^{(1)}(\mathbf{k}_1, \mathbf{k}_2) + (gk/k_1k_2)^{1/4}D^{(2)}(\mathbf{k}_1, \mathbf{k}_2) - (gkk_1k_2)^{1/4}D^{(3)}(\mathbf{k}_1, \mathbf{k}_2)]\,\delta_{0+1+2}, \quad (I-3)$$

$$F^{(1)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = 1/4\,[(k_2k_3/kk_1)^{1/4}E^{(1)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) - (kk_3/k_1k_2)^{1/4}E^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + (kk_1k_2k_3)^{1/4}E^{(3)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)]\,\delta_{0-1-2-3}, \quad (I-4)$$

$$F^{(2)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = 1/4\,[(k_1k_2/kk_3)^{1/4}E^{(1)}(\mathbf{k}_3, \mathbf{k}_2, -\mathbf{k}_1) - (k_2k_3/kk_1)^{1/4}E^{(1)}(-\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + (k_1k_3/kk_2)^{1/4}E^{(1)}(\mathbf{k}_2, -\mathbf{k}_1, \mathbf{k}_3) +$$

$(kk_3/k_1k_2)^{1/4}\{E^{(2)}(\mathbf{k}_2, -\mathbf{k}_1, \mathbf{k}_3)+E^{(2)}(-\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\}-$

$(kk_1/k_2k_3)^{1/4}E^{(2)}(\mathbf{k}_3, \mathbf{k}_2, -\mathbf{k}_1)+$

$(kk_1k_2k_3)^{1/4}\{E^{(3)}(\mathbf{k}_3, \mathbf{k}_2, -\mathbf{k}_1)+E^{(3)}(\mathbf{k}_2, -\mathbf{k}_1, \mathbf{k}_3)+$

$E^{(3)}(-\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\}]\,\delta_{0+1-2-3},$

(I − 5)

$F^{(3)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)=1/4\,[(k_1k_2/kk_3)^{1/4}E^{(1)}(\mathbf{k}_3, -\mathbf{k}_2, -\mathbf{k}_1)$

$-(k_2k_3/kk_1)^{1/4}\{E^{(1)}(-\mathbf{k}_1, -\mathbf{k}_2, \mathbf{k}_3)+E^{(1)}(-\mathbf{k}_1, \mathbf{k}_3, -\mathbf{k}_2)\}$

$+(kk_2/k_1k_3)^{1/4}E^{(2)}(-\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_2)$

$+(kk_1/k_2k_3)^{1/4}E^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\}-$

$(kk_3/k_1k_2)^{1/4}E^{(2)}(-\mathbf{k}_1, -\mathbf{k}_2, \mathbf{k}_3)+$

$(kk_1k_2k_3)^{1/4}\{E^{(3)}(\mathbf{k}_3, \mathbf{k}_2, -\mathbf{k}_1)+E^{(3)}(\mathbf{k}_1, \mathbf{k}_3, -\mathbf{k}_2)+$

$E^{(3)}(-\mathbf{k}_1, -\mathbf{k}_2, \mathbf{k}_3)\}]\,\delta_{0+1+2-3}$

and (I − 6)

$F^{(4)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)=$

$1/4\,[-(k_2k_3/kk_1)^{1/4}E^{(1)}(-\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3)+$

$(kk_3/k_1k_2)^{1/4}E^{(2)}(-\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3)+$

$(kk_1k_2k_3)^{1/4}E^{(3)}(-\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3)]\,\delta_{0+1+2+3},$

(I − 7)

where, $\delta_{0+1-2-3}=\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)$ and

$D^{(1)}(\mathbf{k}_1, \mathbf{k}_2)=\mathbf{k}_1\mathbf{k}_2+k_1^2,$

$$D^{(2)}(\mathbf{k}_1,\ \mathbf{k}_2)=\tfrac{1}{2}(\mathbf{k}_1\mathbf{k}_2-k_1k_2)\ ,$$

$$D^{(3)}(\mathbf{k}_1,\ \mathbf{k}_2)=\tfrac{1}{2}(k_1+k_2)\ ,$$

$$E^{(1)}(\mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3)=\tfrac{1}{2}k_1\{k_1^2+\mathbf{k}_1(\mathbf{k}_2+\mathbf{k}_3)\}\ ,$$

$$E^{(2)}(\mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3)=-\tfrac{1}{2}(\mathbf{k}_1\mathbf{k}_2-k_1k_2)\ \{|\mathbf{k}_1+\mathbf{k}_2|$$
$$-(k_1^2+k_2^2)\}\ ,$$

$$E^{(3)}(\mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3)=-(1/6)\{(k_1+k_2)\ |\mathbf{k}_1+\mathbf{k}_2|$$
$$+(k_2+k_3)\ |\mathbf{k}_2+\mathbf{k}_3|$$
$$+(k_3+k_1)\ |\mathbf{k}_3+\mathbf{k}_1|$$
$$-(k_1^2+k_2^2+k_3^2)\}\ .$$

Appendix II On Canonical Form

As is well known, the energy of deep-water gravity waves is represented by

$$E=\frac{1}{2}\int_S d\mathbf{r}\ [\int_{-B}^{\eta}(\nabla\phi)^2 dz+g\eta^2], \qquad (\text{II}-1)$$

where, S means the total surface considered here, B is the depth where the wave effect diminishes. This expression contains the volume integral over all region occupied with the fluid, it can be replaced by the surface integrals by means of certain transformation of variables as follows.

From the Gauss' theorem, the first term of the right hand side of (II − 1) (KINEMATIC ENERGY) is expressed by the next equation

$$E_1=\frac{1}{2}\int_S \phi\ (\partial\phi/\partial n)\ dS, \qquad (\text{II}-2)$$

when S means the fluid surface $z=\eta\ (\mathbf{r},\ t)$.
By use of the theorem of differential geometry, the relations

$$(\partial\phi/\partial n)_\eta=(\phi_z-\nabla_h\phi\nabla_h\eta)/\{1+(\nabla_h\eta)^2\}^{1/2}\Big|_\eta$$

and $$dS=\{1+(\nabla_h\eta)^2\}^{1/2}\ d\mathbf{r}$$

are derived. Where, the operator ∇_h means the horizontal two-dimensional gradients.

Furthermore, if we consider the kinematic condition

$$\phi_z-\nabla_h\phi\nabla_h\eta=\eta_t$$

at $z=\eta$, (II − 2) is proved to be replaced by

$$E_1=\frac{1}{2}\int_\Sigma \phi^s\ (\partial\eta/\partial t)\ d\mathbf{r}. \qquad (\text{II}-3)$$

If we add the potential energy term to this, total energy H comes out to

$$H=\frac{1}{2}\int_S d\mathbf{r}\ [\phi^s\eta_t+g\eta^2], \qquad (\text{II}-4)$$

where, ϕ^s is the value of the potential at the fluid surface.

An alternative proof of this theorem is proposed by West(1981). The interpretation of H to be explained as the Hamiltonian of water waves accompanied with the canonical variables ϕ^s and η was presented by Miles(1977), Milder(1977).

To the Hamiltonian (II − 4), the new variables p, q are defined by use of the Fourier transform as

$$\phi^s(\mathbf{r}, t) = (2\pi)^{-1} \iint_{-\infty}^{\infty} d\mathbf{k}\, p(\mathbf{k}, t)\, e^{i\mathbf{k}\mathbf{r}} \qquad (\text{II}-5)$$

and

$$\eta(\mathbf{r}, t) = (2\pi)^{-1} \iint_{-\infty}^{\infty} d\mathbf{k}\, q(\mathbf{k}, t)\, e^{i\mathbf{k}\mathbf{r}}. \qquad (\text{II}-6)$$

By use of them, H is represented by p, q as follows

$$H = \tfrac{1}{2} \int_{-\infty}^{\infty} d\mathbf{k} \{p^*(\mathbf{k})\, q_t(\mathbf{k}) + g\, q^*(\mathbf{k})\, q(\mathbf{k})\}. \qquad (\text{II}-7)$$

In order to eliminate the function $q_t(\mathbf{k})$ from (II − 7), we use the equations presented in Stiassnie & Shemer(1984) (they did not discuss the canonical form), that

$$q_t(\mathbf{k}) = w^s +$$

$$\frac{1}{2\pi} \iint_{-\infty}^{\infty} (\mathbf{k}_1 \cdot \mathbf{k}_2)\, d\mathbf{k}_1 d\mathbf{k}_2\, p(\mathbf{k}_1)\, q(\mathbf{k}_2)\, \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)$$

and

$$w^s = k\, p(\mathbf{k}) -$$

$$\frac{1}{2\pi} \iint_{-\infty}^{\infty} k_1 [k - k_1]\, d\mathbf{k}_1 d\mathbf{k}_2\, p(\mathbf{k}_1)\, q(\mathbf{k}_2)\, \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2),$$

where, we adopt the notations used here and truncated the perturbation series up to the term necessary in this discussion. The function w^s means the value of ϕ_z at the fluid surface.

We make q_t the function of p, q and substitute it into (II − 7) and the representation

$$H = \tfrac{1}{2} \int_{-\infty}^{\infty} d\mathbf{k} \{k\, p^*(\mathbf{k})\, p(\mathbf{k}) + g\, q^*(\mathbf{k})\, q(\mathbf{k})\} +$$

$$\frac{1}{4\pi}\iint_{-\infty}^{\infty} K^{(2)} dk\, dk_1\, dk_2\, p^*(k)\; p(k_1)\; q(k_2)\; \delta(k-k_1-k_2)$$

$$-\frac{1}{8\pi^2}\iiint_{-\infty}^{\infty} K^{(3)} dk\, dk_1\, dk_2\, dk_3\, p^*(k)\; p(k_1)\; q(k_2)\; q(k_3)$$

$$\times\,\delta(k-k_1-k_2-k_3) \qquad (\mathrm{II}-8)$$

is derived. The first term is the well-known Hamiltonian of linear wave field. The kernels $K^{(2)}$ and $K^{(3)}$ are

$$K^{(2)}(k,\; k_1,\; k_2) = (k_1 k_2) - k_1(k-k_1)\,, \qquad (\mathrm{II}-9)$$

$$K^{(3)}(k_1,\; k_2,\; k_3) = k_1(k_2 k_3)\,. \qquad (\mathrm{II}-10)$$

Appendix III Randomization of the Zakharov equation

The arguments applied to the randomization of narrow-band wave system by Longuet-Higgins(1976) is extended to this problem of arbitrary band width wave system in the following.

The exact form of the Zakharov equation is presented by (3 − 1) in Chapter 3. Here, we abbrebiate it to the following form

$$i\frac{dB_0}{dt}=\int dK\,T\,B_1^*B_2B_3\delta\,e^{i\Delta t}. \qquad (\text{III}-1)$$

Multiplying B_0^* to the both sides of the equation and subtracting it from its complex conjugate, we have

$$i\frac{d|B_0|^2}{dt}=2i\,\mathrm{Im}\int dK\,T\,B_1^*B_2B_3B_0^*\delta\,e^{i\Delta t}. \qquad (\text{III}-2)$$

If we write the ensemble average of $|B_0|^2$ by $<|B_0|^2>=C_0$, we get from (III − 2) the statistical equation

$$i\frac{dC_0}{dt}=2i\,\mathrm{Im}\int dK\,\tilde{T}\,C_0C_1. \qquad (\text{III}-3)$$

In this equation, the right-hand side is 0, because all the quantities in the integrant are real numbers. Therefore, the energy spectrum of the stochastic wave field does not vary to the 4-th order.

Finally, time derivative of the 4-th order mutual products are calculated as

$$i\,(B_0^*B_1^*B_2B_3)_t=i\,B_{0t}^*B_1^*B_2B_3+i\,B_0^*B_{1t}^*B_2B_3$$

$$+i\,B_0^*B_1^*B_{2t}B_3+i\,B_0^*B_1^*B_2B_{3t} \qquad (\text{III}-4)$$

Substituting (III − 1) to the time derivatives in the right hand side of (III − 4), averaging the whole equation, and remaining up to the 6-th order of magunitude B, it is yielded as

$$i<(B_0^*B_1^*B_2B_3)>_t=2T\{C_2C_1C_0+C_3C_1C_0$$

$$-C_0C_2C_3-C_1C_2C_3\}\,\delta\exp(-i\Delta t). \qquad (\text{III}-5)$$

Rewriting (Ⅲ－2), we get the equation

$$i\frac{dC_0}{dt}=-2i\mathrm{Re}\int dK\,T\,i<B_1^*B_2B_3B_0^*>\delta\,e^{i\Delta t} \qquad (\text{Ⅲ}-6)$$

To evaluate the right hand side of this equation, (Ⅲ－5) is integrated to be

$$i<(B_0^*B_1^*B_2B_3)>=2\int_{-\infty}^{t}d\tau\,T\,\{\}\,\delta\,e^{-i\Delta\tau} \qquad (\text{Ⅲ}-7)$$

In this equation, {} denotes the quantity in the bracket in (Ⅲ－5). Therefore,

$$i\frac{dC_0}{dt}=-4i\mathrm{Re}\int dK\,T^2\,\{\}\,\delta\int_{-\infty}^{t}d\tau\;e^{i\Delta(t-\tau)} \qquad (\text{Ⅲ}-8)$$

is obtained.
The last definite integral is turn out to be $\pi\delta(\Delta)$, so that the final result has the form

$$\frac{dC_0}{dt}=4\pi\int dK\,T_{0123}^2\,\{C_2C_3(C_0+C_1)-C_0C_1(C_2+C_3)\}\,\delta_{0123}\,\delta(\Delta_{0123}), \qquad (\text{Ⅲ}-9)$$

which is just the same form with the energy transport equation among the spectral components first given by Hasselmann(1962, 1963a, 1963b).

AppendixⅣ. Narrow band approximation of the Zakharov equation

So called non-linear Schroedinger equation was first derived in the paper of Zakharov(1968) himself. In this Appendix, we interprete the procedure in terms of the symbols used in this paper.

We restrict ourselves that B has large value only in the vicinity of certain central wave-number $\mathbf{k}_0$. That is, $\mathbf{k}=\mathbf{k}_0+\Psi$, so

$$B(\mathbf{k})\exp\{i(\mathbf{k}\mathbf{r}-\omega t)\}=B(\mathbf{k})\exp\{i(\mathbf{k}_0\mathbf{r}+\Psi\mathbf{r}-\omega t+\omega_0 t+\omega_0 t)\}$$

$$\equiv A(\Psi)\exp(i\Psi\mathbf{r})\exp\{i(\mathbf{k}_0\mathbf{r}-\omega_0 t)\} \qquad (\text{IV}-1)$$

is introduced to the fractional wave-number Ψ. Using this formula, the elevation η is expressed as

$$\eta=\left[\frac{1}{2\pi}(k_0/2\omega_0)^{\frac{1}{2}}\int_{-\infty}^{\infty}d\Psi A(\Psi)e^{i\Psi\mathbf{r}}\right]e^{i(\mathbf{k}_0\mathbf{r}-\omega_0 t)}+c.c. \qquad (\text{IV}-2)$$

By definition, the quantity in 【】 is one half of the wave envelope $a(\mathbf{r}, t)$, therefore, the relation of A to a is

$$a(\mathbf{r}, t)=\frac{1}{2\pi}(k_0/2\omega_0)^{\frac{1}{2}}\int_{-\infty}^{\infty}d\Psi A(\Psi)e^{i\Psi\mathbf{r}}. \qquad (\text{IV}-3)$$

On the other hand, (Ⅳ－1) is substituted to the Zakharov equation (Ⅲ－1) to be

$$i\frac{dA_0}{dt}-(\omega-\omega_0)A_0=\int dK\,T A_1^* A_2 A_3 \delta e^{i\Delta t}. \qquad (\text{IV}-4)$$

Assuming that the band width $\Psi=(\phi, \lambda)$ is sufficiently small, we expand the dispersion relation $\omega=f(k)$ around ω_0 up to the 2-nd order of ϕ, λ to have

$$\omega - \omega_0 = (\omega_0 / 2k_0)\,\phi - (\omega_0 / 8k_0^2)\,\phi^2 + (\omega_0 / 4k_0^2)\,\lambda^2 . \qquad (\text{IV}-5)$$

Substituting this into (IV－4) and Fourier transforming with respect to $\mathbf{k}$, we get the final equation by use of the approximate kernel $T_{0123} = T_{0000} = k_0^3 / 4\pi^2$

$$i\left(\frac{\partial a}{\partial t} + c_g \frac{\partial a}{\partial x}\right) + \frac{\omega_0}{8k_0^2}\frac{\partial^2 a}{\partial x^2} - \frac{\omega_0}{4k_0^2}\frac{\partial^2 a}{\partial y^2} = \frac{\omega_0 k_0^2}{2}|a|^2 a . \qquad (\text{IV}-6)$$

This equation agrees with the 2-dimensional Nonlinear Schroedinger equation for deep-water gravity waves (see, Yuen & Lake(1982)) .

Modifications of (IV－6) for including the mean flow effects were proposed by Dysthe(1979) to deep-water waves and by Tomita(1985b, 1986) to waves on a finite water depth.

Appendix V Kernel T of the Zakharov equation

The third order interaction coefficient $T(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ appearing in (3 − 1) was first found by Zakharov(1968) and rederived by Crawford et.al.(1981) is exhibited below with some minor misprints removed:

$$T(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = T_{0123} =$$

$$- \frac{2V^{(-)}_{3.3-1.1} V^{(-)}_{0.2.0-2}}{\omega_{1-3} - \omega_3 + \omega_1} - \frac{2V^{(-)}_{2.0.2-0} V^{(-)}_{1.1-3.3}}{\omega_{1-3} - \omega_1 + \omega_3}$$

$$- \frac{2V^{(-)}_{2.2-1.1} V^{(-)}_{0.3.0-3}}{\omega_{1-2} - \omega_2 + \omega_1} - \frac{2V^{(-)}_{3.0.3-0} V^{(-)}_{1.1-2.2}}{\omega_{1-2} - \omega_1 + \omega_2}$$

$$- \frac{2V^{(-)}_{0+1.0.1} V^{(-)}_{2+3.2.3}}{\omega_{2+3} - \omega_2 + \omega_3} - \frac{2V^{(+)}_{-2-3.2.3} V^{(+)}_{0.1.-0-1}}{\omega_{2+3} - \omega_2 + \omega_3}$$

$$+ W_{0.1.2.3} ,$$

where,

$$V^{(\pm)}_{0.1.2} = \frac{1}{8\pi\sqrt{2}} \{ (\mathbf{k}_0 \cdot \mathbf{k}_1 \pm k_0 k_1) \left[\frac{\omega_0 \omega_1}{\omega_2} \frac{k_2}{k_0 k_1} \right]^{1/2}$$

$$+ (\mathbf{k}_0 \cdot \mathbf{k}_2 \pm k_0 k_2) \left[\frac{\omega_0 \omega_2}{\omega_1} \frac{k_1}{k_0 k_2} \right]^{1/2}$$

$$+ (\mathbf{k}_1 \cdot \mathbf{k}_2 + k_1 k_2) \left[\frac{\omega_1 \omega_2}{\omega_0} \frac{k_0}{k_1 k_2} \right]^{1/2} \} ,$$

$$\omega_{1+2} = \omega(\mathbf{k}_1 + \mathbf{k}_2)$$

and

$$W_{0.1.2.3} = \overline{W}_{-0.-1.2.3} + \overline{W}_{2.3.-0.-1} - \overline{W}_{2.-1.-0.3} - \overline{W}_{-0.2.-1.3} +$$

$$\overline{W}_{-0.302.-1} - \overline{W}_{3.-1.2.-0}$$

with

$$\bar{W}_{0,1,2,3} = \frac{1}{64\pi^2}\left[\frac{\omega_0\omega_1}{\omega_2\omega_3} \; k_0 k_1 k_2 k_3\right]^{1/2} \times$$

$$\{2(k_0 + k_1) - k_{1+3} - k_{1+2} - k_{0+3} - k_{0+2}\}$$

and $$k_{1+2} = |\mathbf{k}_1 + \mathbf{k}_2| \; .$$

Appendix VI Dispersion Relation in Tertiary Resonant Interaction

In general, the kernel $T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ of the Zakharov equation is so complicated that the simple analytical expression was obtained only in the cases that $\mathbf{k}_1=\mathbf{k}_2=\mathbf{k}_3=\mathbf{k}_4$ (single wave) and $\mathbf{k}_1=-\mathbf{k}_2=-\mathbf{k}_3=\mathbf{k}_4$ (standing wave) in the paper by Okamura(1984). We deal with here the next simplest case that $\mathbf{k}_1=\mathbf{k}_3=k(\cos\theta, \sin\theta)$ and $\mathbf{k}_2=\mathbf{k}_4=k(\cos\theta, -\sin\theta)$.

If there exists only two trains of wave of exactly same amplitude a and wavelength $\lambda=2\pi/k$ intersecting by the angle 2θ, the equations corresponding to (3 − 3) become

$$i\frac{dB_1}{dt} = \{T_{1111}B_1^*B_1+\widetilde{T}_{1221}B_2^*B_2\}\ B_1 \qquad (\text{VI}-1-1)$$

and

$$i\frac{dB_2}{dt} = \{\widetilde{T}_{2112}B_1^*B_1+T_{2222}B_2^*B_2\}\ B_2\ . \qquad (\text{VI}-1-2)$$

These are easily solved by setting $B_1=b\exp(-i\chi_1)$, $B_2=b\exp(-i\chi_2)$ with real constant b. From (VI − 1) and (3 − 5), $\chi_{1,2}$ are given by

$$\chi_1=\{T_{1111}+\widetilde{T}_{1221}\}\ (2\pi^2\omega/k)\ a^2 \qquad (\text{VI}-2-1)$$

and

$$\chi_2=\{\widetilde{T}_{2112}+T_{2222}\}\ (2\pi^2\omega/k)\ a^2\ . \qquad (\text{VI}-2-2)$$

The resultant of the two waves are called the SHORT CRESTED WAVE of amplitude $A=2a$ and its dispersion relation was derived by Mollo-Christensen(1981) that

$$\omega=\omega_0\ \{1+\tfrac{1}{4}A^2k^2F(\theta)\} \qquad (\text{VI}-3)$$

where

$$F(\theta)=(8\cos^2\theta-3-2\cos^4\theta)/2+\sin^2\theta(\cos\theta+2-4\cos^2\theta)/(2-\cos\theta). \qquad (\text{VI}-4)$$

These formula are also verified from the equation (2 − 8) of Longuet-Higgins & Phillips(1962), after some minor correction.

We can rederive this result by means of Zakharov theory that

$$T_{1111}+\widetilde{T}_{1221}=\widetilde{T}_{2112}+T_{2222}=(k^3/2\pi^2)\ F(\theta) \qquad (\text{VI}-5)$$

after some algebraic manipulation. At least in these simple cases, it is revealed that the Zakharov theory yields an identical result with the classical one (see Tomita(1985a)).

Appendix Ⅶ Conservation laws of the Zakharov equation

Here, we define two quantities E and C such that

$$E_i = g A_i^2 / 2 = \omega_i B_i B_i^* / (2\pi)^2 \qquad (\text{Ⅶ}-1)$$

$$C_i = E_i / \omega_i = B_i B_i^* / (2\pi)^2 \qquad (\text{Ⅶ}-2)$$

They are called the energy and the wave action of waves. Considering the total energy of three waves that

$$E = E_1 + E_2 + E_3 = \{\omega_1 B_1 B_1^* + \omega_2 B_2 B_2^* + \omega_2 B_2 B_2^*\} / (2\pi)^2 ,$$

we obtain the following expression to its derivative dE/dt by use of (3－3－1) ～ (3－3－3),

$$i\frac{dE}{dt} = \{\omega_1 (\tilde{T}_{1123} e^{i\Delta t} B_1^{*2} B_2 B_3 - c.c.) +$$

$$\omega_2 (T_{2311} e^{-i\Delta t} B_1^2 B_2^* B_3^* - c.c.) +$$

$$\omega_3 (T_{2311} e^{-i\Delta t} B_1^2 B_2^* B_3^* - c.c.)\} / (2\pi)^2$$

$$= [\{\omega_1 \tilde{T}_{1123} - \omega_2 T_{2311} - \omega_3 T_{3211}\} e^{i\Delta t} B_1^{*2} B_2 B_3 - c.c.] / (2\pi)^2,$$

where c.c. signifies the complex conjugate of preceding term.

Because the equalities $\tilde{T}_{1123} = 2T_{2311} = 2T_{3211} = 2T$ holds with the resonance condition $2\omega_1 - \omega_2 - \omega_3 = 0$, the change of energy is

$$\frac{dE}{dt} = \mathrm{Im}[\{2\omega_1 - \omega_2 - \omega_3\} T e^{i\Delta t} B_1^{*2} B_2 B_3] / (2\pi)^2 = 0 . \qquad (\text{Ⅶ}-3)$$

Thus, the total energy conservation is proved in the present situation. The conservation of total wave action C is also derived by the method akin to the above procedure. It is in contrast to the case of capillary-gravity waves (for reference Leibovich & Seebas(1974) or Whitham(1974)) .The conservation of wave action leads to next simultaneous equations with respect to the amplitudes A_i,

$$g A_1^2 / \omega_1 + g A_2^2 / \omega_2 + g A_3^2 / \omega_3 = \text{const}$$

and

$$g A_2^2 / \omega_2 - g A_3^2 / \omega_3 = \text{const} \quad .$$

Initial values of $A_1 = A_{10}$, $A_2 = A_{20}$, $A_3 = 0$ are substituted to the right-hand constant terms and the elimination of A_2 leads to

$$A_1^2 / \omega_1 + 2 A_3^2 / \omega_3 = A_{10}^2 / \omega_1 \quad .$$

It means that the capable maximum amplitude of tertiary wave has a limit

$$A_3 \leqq (\omega_3 / 2 \omega_1)^{1/2} A_{10} = (2\gamma - 1/2\gamma)^{1/2} A_{10} = 0.844 A_{10} \quad . \tag{VII-4}$$

In order to make the condition of validity of the Zakharov equation clear, we estimate (1-10) with respect to γ. The condition

$$\Delta = 2 \omega_1 - \omega_2 - \omega_3 \sim 0 \tag{VII-5}$$

is to be evaluated. This is rewritten as

$$\Delta = \omega_0 - \omega_3 \sim \varepsilon^2 \omega_3 \tag{VII-6}$$

using $\omega_0 = 2 \omega_1 - \omega_2$. Small quantity Δ is estimated that

$$\Delta = (g k_0)^{1/2} - (g k_3)^{1/2}$$

$$= -\tfrac{1}{2} (g / k_3)^{1/2} (k_0 - k_3) = \tfrac{1}{2} \omega_3 2 \delta k .$$

By virtue of (1-11), we see that $2\delta k = -\beta k_3 \delta \gamma$ where $\delta \gamma = \gamma - \gamma_0$. Substituting it to the relation,

$$\Delta = -\tfrac{1}{2} \omega_3 \beta \delta \gamma \tag{VII-7}$$

is yielded. Thus, from (VI-6), $\frac{1}{2} \beta \mid \delta \gamma \mid \sim \varepsilon^2$ results. Using the constant value $\beta = 0.497 \sim 0.5$ from the theory and adopting the small quantity $\varepsilon \sim A k$ to be 0.2 from the experiment, we obtain an estimation of γ such that

$$|\delta\gamma| \sim 0.16, \quad \text{i.e.,} \quad 1.58 < \gamma < 1.90 \qquad (\text{VII}-8)$$

Although the estimation examined above is not always accurate, the range of γ is verified almost to cover that of experiment in this paper.

AppendixⅧ On the similarity of the theories

The Zakharov equation in Chapter 3 has the dimensional form and the calculations are conducted to the wavelengths comparable with the magnitude used in the experiment. However, all the results obtained in this paper must be applicable to the scale of actual ocean. In order to show this, we derive the non-dimensional form of the Zakharov equation and the classical solution.

First, we see from (3 − 2) and (3 − 3) the dimensions of B and T to be $B = [m^{3/2} s^{-1/2}]$, $T = [m^{-3}]$. Thus, we introduce the non-dimensional variables such as:

$$\omega_n = \omega_R w_n \ , \qquad (\text{Ⅷ} - 1 - 1)$$

$$\omega_R t = \tau \ , \qquad (k_R = \omega_R^2 / g) \ , \qquad (\text{Ⅷ} - 1 - 2)$$

$$T_{1234} = T_R U_{1234} \ , \qquad (\text{Ⅷ} - 1 - 3)$$

$$(T_R = T\ (\mathbf{k}_R, \mathbf{k}_R, \mathbf{k}_R, \mathbf{k}_R) = k_R^3 / 4\pi^2) \ ,$$

$$B_n = B_R F_n \ , \qquad (\text{Ⅷ} - 1 - 4)$$

$$(B_R = (2\omega_R / k_R)^{1/2} A_R) \ .$$

Substituting them in (3 − 3 − 1) for example, we have a non-dimensional form of the equation

$$i \frac{dF_1}{dt} = \mu\ [\ \{U_{1111} F_1^* F_1 + U_{1221} F_2^* F_2 + U_{1331} F_3^* F_3\}\ F_1 + \widetilde{U}_{1123} e^{i\delta\tau} F_1^* F_2 F_3] \qquad (\text{Ⅷ} - 2)$$

where, $\delta = w_1 + w_2 + w_3 + w_4$ and $\mu = \omega_R^{-1} B_R^2 T_R$ is a non-dimensional constant. Connecting the relations (Ⅷ − 1) together, coefficient μ is estimated as

$$\mu = \tfrac{1}{2}\ (A_R k_R)^2 \ . \qquad (\text{Ⅷ} - 3)$$

This is nothing but a wave steepness.

By the similar manner, the classical solution (2 − 3) is