

Chapter 3

Formulation of second order forces due to Volterra functional series and Application of Wiener's filter theory

3.1 Relationship between Volterra functional series and second order force system

Let $\vec{F}(t)$ denote the nonlinear total second order force vector of a floating structure to a random excitation $\{\zeta(t) \mid t \in R\}$. Since $\vec{F}(t)$ is the response vector to the entire time history of $\zeta(t)$, we call $\vec{F}(t)$ a functional vector defined on a class of excitation $\zeta(t)$ as

$$\vec{F}(t) = \vec{F}[\zeta(t)] \quad (3.1)$$

If $\vec{F}[\zeta(t)]$ is a continuous functional vector of $\zeta(t)$ in the function space sense, then $\vec{F}(t)$ can be expanded in a functional vector power series such that

$$\begin{aligned} \vec{F}(t) &= \vec{F}_0 + \int \vec{h}_1(t, t_1) \zeta(t_1) dt_1 + \dots \\ &+ \int \dots \int \vec{h}_n(t, t_1, \dots, t_n) \zeta(t_1) \dots \zeta(t_n) dt_1 \dots dt_n + \dots \end{aligned} \quad (3.2)$$

If this series represents a causal physical system, then the kernel function vectors satisfy

$$\vec{h}_n(t, t_1, \dots, t_n) = \vec{0} \quad t_i > t \quad (3.3)$$

Series satisfying this property were studied by Volterra¹⁾, and series of the form (3.1) that satisfy Eq.(3.3) are called Volterra functional vector series.

If the nonlinear system is time invariant, then kernel function vectors in Eq.(3.2) depend only on time difference. Thus,

$$\begin{aligned} \vec{F}(t) &= \vec{F}_0 + \int \vec{g}_1^f(\tau) \zeta(t - \tau) d\tau + \dots \\ &+ \int \int \vec{g}_2^f(\tau_1, \tau_2) \zeta(t - \tau_1) \zeta(t - \tau_2) d\tau_1 d\tau_2 + \dots \end{aligned} \quad (3.4)$$

where \vec{F}_0 is a constant vector. In general, the kernel function vectors in Eq.(3.4) may not be symmetric for their arguments. However, a permutation of indices in any kernel vectors only affects the order in which the integration is carried out but does not affect the response. Thus, for the purpose of analysis, symmetric kernel vector may be assumed without loss of generality; i.e.

$$\vec{g}_n(\tau_1, \tau_2, \dots, \tau_n) = \frac{1}{n!} \sum_{[i]} \vec{g}_n^f(\tau_{i1}, \dots, \tau_{in}) \quad (3.5)$$

If the kernels are continuous and absolutely integrable and if the input is bounded and the contribution from terms of order n in Eq.(3.4) decreases to zero as $n \rightarrow \infty$, then it can be proved that the functional power series (3.4) converge uniformly.

We shall limit our analysis to excitation effects through second order and $\vec{F}_0 = 0$. Then Eq.(3.4) is truncated at $n=2$ and takes the following form:

$$\vec{F}(t) = \int \vec{g}_1^f(\tau) \zeta(t - \tau) d\tau + \int \int \vec{g}_2^f(\tau_1, \tau_2) \zeta(t - \tau_1) \zeta(t - \tau_2) d\tau_1 d\tau_2 \quad (3.6)$$

And we will treat the vector function as the scalar function hereinafter for simplicity. If $\zeta(t)$ is a wave excitation, this series can be used to analyze the response that is proportional to both the wave height and the squared wave height. There exist the time and spatial dependencies in the incident wave system. But since the wave system have a dispersivity, it is not necessary to consider the spatial dependency as indicated by Hasselmann²⁾. It may also be mentioned that the adopted formulation is consistent with second order corrections to a linear wave field, in the sense that such corrections may be incorporated in Eq.(3.6) where $\zeta(t)$ then denotes the linear part of the wave field. Consequently, the assumption that $\zeta(t)$ is a linear, Gaussian wave process is consistent with the second order model in Eq.(3.6). If the kernels in Eq.(3.6)

are continuous and absolutely integrable, then the kernels possess the Fourier transform. The transform pairs are defined as follows:

$$\begin{aligned}
 g_1^f(t) &= \frac{1}{2\pi} \int_{\tau} G_2^f(\omega) \exp(i\omega\tau) d\omega \\
 G_1^f(\omega) &= \int_{\omega} g_1^f(\tau) \exp(-i\omega\tau) d\tau \\
 g_2^f(\tau_1, \tau_2) &= \frac{1}{4\pi^2} \int_{\omega_1} \int_{\omega_2} G_2^f(\omega_1, \omega_2) \exp\{i(\omega_1\tau_1 + \omega_2\tau_2)\} d\omega_1 d\omega_2 \\
 G_2^f(\omega_1, \omega_2) &= \int_{\tau_1} \int_{\tau_2} g_2^f(\tau_1, \tau_2) \exp\{-i(\omega_1\tau_1 + \omega_2\tau_2)\} d\tau_1 d\tau_2
 \end{aligned} \tag{3.7}$$

In Eq.(3.7) the kernel g_1^f is a linear impulse response function, and its transform, G_1^f , is a linear transfer function. The kernel g_2^f is analogous to the linear impulse response function and is called "*quadratic impulse response function*". Its transform, G_2^f , is called "*quadratic transfer function*". Since the kernel $g_2^f(\tau_1, \tau_2)$ can be assumed to be symmetrical in its arguments; i.e.

$$g_2^f(\tau_1, \tau_2) = g_2^f(\tau_2, \tau_1) \tag{3.8}$$

thus

$$G_2^f(\omega_1, \omega_2) = G_2^f(\omega_2, \omega_1) \tag{3.9}$$

Consequently, the quadratic transfer function is symmetrical about the line $\omega_1 = \omega_2$ in the (ω_1, ω_2) plane.

If $\zeta(t)$ is a Gaussian random wave with one-sided spectrum U , Rice³⁾ has shown that it is represented in the following stochastic integral:

$$\zeta(t) = \int \cos(\omega t - \mu(\omega)) \sqrt{2U(\omega)} d\omega \tag{3.10}$$

where μ is a random phase distributed uniformly over 0° to 360° . This representation means the stochastic integral, and it converges in the sense of stochastic quadratic mean.

Substituting (3.10) into (3.6) we have:

$$F^{(1)}(t) = \int \cos(\omega t - \mu(\omega) + \theta_1(\omega)) \sqrt{2 |G_1^f(\omega)|^2 U(\omega)} d\omega \tag{3.11}$$

$$\begin{aligned}
 F^{(2)}(t) &= \iint \cos\{(\omega_1 + \omega_2)t - (\mu(\omega_1) + \mu(\omega_2)) + \theta_2(\omega_1, \omega_2)\} \\
 &\quad \times \sqrt{|G_2^f(\omega_1, \omega_2)|^2 U(\omega_1)U(\omega_2)} d\omega_1 d\omega_2 \\
 &+ \iint \cos\{(\omega_1 - \omega_2)t - (\mu(\omega_1) - \mu(\omega_2)) + \theta_2(\omega_1, -\omega_2)\} \\
 &\quad \times \sqrt{|G_2^f(\omega_1, -\omega_2)|^2 U(\omega_1)U(\omega_2)} d\omega_1 d\omega_2
 \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} G_1^f(\omega) &= |G_1^f(\omega)| \exp(i\theta_1(\omega)) \\ G_2^f(\omega_1, \omega_2) &= |G_2^f(\omega_1, \omega_2)| \exp\{i\theta(\omega_1, \omega_2)\} \end{aligned}$$

It is clear that the first term on right hand side of (3.12) shows the sum component of second order force and the second term indicates the difference component. Taking the ensemble average of Eq.(3.12), and taking into account a statistical independence of the random phases, we get:

$$E[F^{(2)}] = \int G_2^f(\omega, -\omega)U(\omega)d\omega \quad (3.13)$$

While the time average of $\overline{F}^{(2)}(t)$ is represented in the following form:

$$\overline{F}^{(2)} = \sum_i F_d(\omega_i) |a_i|^2 \quad (3.14)$$

By using the relationship as;

$$a_i = \sqrt{2U(\omega_i)d\omega_i}$$

Eq.(3.14) can be expressed in the following integral:

$$\overline{F}^{(2)} = \int 2F_d(\omega)U(\omega)d\omega \quad (3.15)$$

If (3.13) is equal to (3.15), the following relationship holds:

$$G_2^f(\omega, -\omega) = 2F_d(\omega) \quad (3.16)$$

Similarly, from comparison between (3.12) and (2.11) the system function G_2^f can be related to the transfer function of slowly varying drift force like:

$$G_2^f(\omega_1, -\omega_2) = 2f_2(\omega_1, -\omega_2) \quad (3.17)$$

3.2 Application of Wiener's filter theory to slowly varying drift force

It is clear from (3.12) that the slowly varying drift force can be expressed by a quadratic form of random processes. So we expect that the quadratic impulse response function may reveal a kind of filter function in the field of communication engineering. Thus if the system considered is Ergodic, it is possible from the Wiener's theory to replace g_2^f by a optimum linear filter, i.e.:

$$F^{(2)}(t) = \int w_2(\tau)\zeta^2(t-\tau)d\tau \quad (3.18)$$

where w_2 is an optimum linear impulse response function.

The Wiener's theory⁴⁾ provides an optimum filter function w_2 under the following three conditions:

- (1) The input process must be an Ergodic process and its spectral density can be resolved into factors.
- (2) A criterion of error minimizes the least mean square of error.
- (3) A filter function is linear and causal.

The criterion of error between the second term of Eq.(3.6) and Eq.(3.18), that is, J can be obtained from the conditions (1) and (3) as follows:

$$J = E\left[\left\{\iint g_2^f(\tau_1, \tau_2)\zeta(t - \tau_1)\zeta(t - \tau_2)d\tau_1d\tau_2 - \int w_2(\tau)\zeta^2(t - \tau)dt\right\}^2\right] \quad (3.19)$$

The problem minimizes J in (3.19) with respect to an arbitrary function w_2 , i.e. a kind of stationary value problems in calculus of variation.

Let $J[w_2]$ be a functional.

Now, assuming that w_2^0 is a function minimizing J , then the necessary condition for w_2^0 to be an optimum Wiener filter is given by:

$$\lim_{\epsilon \rightarrow 0} \frac{\partial J[w_2^0 + \epsilon w_2]}{\partial \epsilon} = 0 \quad (3.20)$$

This representation is equivalent to the following equation:

$$\begin{aligned} & \int_{\tau_1} \int_{\tau_2} [R_\zeta(0)R_\zeta(\tau_2 - \tau_1) + 2R_\zeta(\tau - \tau_2)R_\zeta(\tau - \tau_2)] \\ & \times \{g_2^f(\tau_1, \tau_2) - w_2(\tau_1)\delta(\tau_2 - \tau_1)\}d\tau_1d\tau_2 = 0 \end{aligned} \quad (3.21)$$

where R_ζ is the auto correlation function of $\zeta(t)$.

Thus we have:

$$g_2^f(\tau_1, \tau_2) = w_2(\tau_1)\delta(\tau_2 - \tau_1) \quad (3.22)$$

If the Fourier transform of w_2 is given by W_2 , the following relation is satisfied.

$$G_2^f(\omega_1, \omega_2) = W_2(\omega_1 + \omega_2) \quad (3.23)$$

Multiplying the incident wave spectrum in both sides of (3.23) and integrating in frequency domain, a concrete form of W_2 is given by:

$$W_2(\omega) = \frac{1}{\sigma_\zeta^2} \int G_2^f(\omega - \omega', \omega')S_\zeta(\omega')d\omega' \quad (3.24)$$

Now, we assume that $G_2^f(\omega_i, -\omega_j)$ is smooth with respect to ω_i and ω_j and that $\frac{\partial G_2^f}{\partial \omega_j} = vG_2^f(\omega_i, -\omega_j)$, $i \neq j$ and v is any small quantity; that is, the tangent planes of $G_2^f(\omega_i, -\omega_j)$ makes small angles with $G_2^f(\omega_i, -\omega_j)$.

Triantafylou⁵⁾ has pointed out that this assumption is valid only for the case that the second order waves need not be considered as shallow water waves. If this is valid, then from Taylor expansion we get:

$$\frac{G_2^f(\omega - \omega', \omega') - G_2^f(\omega', -\omega')}{G_2^f(\omega', -\omega')} \sim \sum \frac{v^n}{n!} \omega^n \quad [\omega \rightarrow \infty]$$

From definition of asymptotic series

$$G_2^f(\omega - \omega', \omega') \sim G_2^f(\omega', -\omega') \cdot \vartheta(\omega) \quad [\omega \rightarrow \infty] \quad (3.25)$$

where $\vartheta(\omega)$ is a response function, of which amplitude exponentially decreases with an increase of ω . This expression is equivalent to the approximation suggested by Newman (see Eq.(2.13)).

Substituting (3.25) into (3.24) we get:

$$W_2(\omega) = \frac{\overline{F}^{(2)}}{\sigma_\zeta^2} \cdot \vartheta(\omega) \quad (3.26)$$

where $\overline{F}^{(2)}$ is the steady drift force in irregular waves (see Eq.(3.13)). Thus, the impulse response function can be expressed as:

$$w_2(\tau) = \frac{1}{2\pi\sigma_\zeta^2} \overline{F}^{(2)} \int \vartheta(\omega) \exp(i\omega\tau) d\omega \quad (3.27)$$

Taking into account that $\vartheta(\omega)$ is an exponential decaying function, it means that w_2 represents a low pass filter function. That is, we can generate the slowly varying drift force by passing $\frac{\overline{F}^{(2)}}{\sigma_\zeta^2} \cdot \zeta^2(t)$ through a low pass filter.

3.3 Estimation of transfer functions of first and second order forces

This section shows the method to estimate transfer functions of first and second order responses from experiments.

If a surface elevation $\zeta(t)$ is expressed by a Gaussian random process with zero mean, the cross correlation function between second order force F and ζ can be represented in the following form:

$$\begin{aligned} R_{F\zeta}(t) &= E[(F(t) - \overline{F})\zeta(t - \tau)] \\ &= \int g_1^f(t_1) R_\zeta(t_1 - t) dt_1 \end{aligned} \quad (3.28)$$

And from Wiener-Khinchine relationship the cross spectrum is given by:

$$S_{F\zeta}(\omega) = G_1^f(\omega) S_\zeta(\omega) \quad (3.29)$$

where S_ζ is a two sided wave spectrum.

This result means that the cross spectrum involves only the first term of the functional polynomials, and the linear transfer function G_1^f is derived by standard cross spectral technique.

Next, we consider a third order moment function as follows:

$$R_{\zeta\zeta F}(\tau_1, \tau_2) = E[\zeta(t + \tau_1)\zeta(t - \tau_1)\{F(t - \tau_2) - \bar{F}\}] \quad (3.30)$$

Substituting (3.6) and taking into account of the symmetry of the quadratic impulse response function g_2^f , Eq.(3.30) becomes:

$$R_{\zeta\zeta F}(\tau_1, \tau_2) = 2 \iint g_2^f(t_1, t_2) R_\zeta(t_1 + \tau_1 + \tau_2) R_\zeta(t_2 - \tau_1 + \tau_2) dt_1 dt_2 \quad (3.31)$$

And utilizing Parseval's formula, the representation in frequency domain is obtained in the following form:

$$R_{\zeta\zeta F}(\tau_1, \tau_2) = 2 \iint G_2^{f*}(\omega_1, \omega_2) S_\zeta(\omega_1) S_\zeta(\omega_2) \\ \times \exp[i\{(\omega_1 - \omega_2)\tau_1 + (\omega_1 + \omega_2)\tau_2\}] d\omega_1 d\omega_2 \quad (3.32)$$

Tick⁶⁾ has defined a cross bispectrum $C_{\zeta\zeta F}$ as a two dimensional Fourier transform of a third order moment function $R_{\zeta\zeta F}$ as follows:

$$R_{\zeta\zeta F}(\tau_1, \tau_2) = \iint \exp\{i(\Omega_1\tau_1 + \Omega_2\tau_2)\} C_{\zeta\zeta F}(\Omega_1, \Omega_2) d\Omega_1 d\Omega_2 \quad (3.33)$$

$$C_{\zeta\zeta F}(\Omega_1, \Omega_2) = \frac{1}{4\pi^2} \iint \exp\{-i(\Omega_1\tau_1 + \Omega_2\tau_2)\} R_{\zeta\zeta F}(\tau_1, \tau_2) d\tau_1 d\tau_2 \quad (3.34)$$

From (3.32) and (3.34) we can find the relationship between the cross bispectrum and the quadratic transfer function in the following form:

$$G_2^f(\omega_1, \omega_2) = \frac{C_{\zeta\zeta F}^*(\omega_1 - \omega_2, \omega_1 + \omega_2)}{S_\zeta(\omega_1) S_\zeta(\omega_2)} \quad (3.35)$$

The method for estimating the cross bispectrum by using experimental data is indicated in Appendix B.

3.4 Comparisons between experimental results and numerical simulations

3.4.1 Model tests

(1) Model

In the experiments an offshore floating structure model supported by twelve legs with footing was used. The configuration of the model and the direction

of incident waves are shown in Fig.3.1. The principal dimensions are indicated in Table 3.1. This is the 1/14.3 scale model of the structure used in the at-sea experiment being carried out in Yura port of Yamagata prefecture.

(2) Test set-up and Measuring items

The model experiments were carried out at the Mitaka No.2 Tank (Length is 400m, the breadth 18m, and the depth 8m) in Ship Research Institute. The model set-up is shown in Fig.3.2. As shown in this figure, the model was restrained by two soft springs through the device which restricted the yaw motion. The spring constant of them was 1.683 kg/m, (0.663 ton/m for the actual structure.).

The measured items are as follows:

- (i) Surge and heave motion measured by a non-contact optical motion measuring system;
- (ii) Pitch motion measured by a vertical gyroscope;
- (iii) Surface elevation measured by a servo needle wave probe fixed at a position, the x coordinate of which is equal to that of the centre of gravity of the model in still water.

(3) Kinds and methods of model tests

(a) Free oscillation test in still water

The natural periods and equivalent damping coefficients in surge motion was obtained from this test. Two kinds of spring coefficients were used. The one was 1.683 kg/m, and the other 5.09 kg/m.

(b) Forced surge sinusoidal or random oscillation tests in still water

Forced surge sinusoidal oscillation tests were carried out at the range of 3.75 to 15 cm in amplitude, and the oscillation period of 17sec. The range corresponds to Keulegan-Carpenter numbers(K_c number) of 1.6 to 6.2. This test was done to study the dependence of the drag force to K_c numbers.

Irregular forced oscillation tests were made to compare with the results of the sinusoidal forced oscillation. Irregular signals for the forced oscillation tests were the surge response data recorded in the following test (d).

(c) Test for measuring steady drift force

Four kinds of tests in regular waves were carried out. Encounter angles of the tests are 0, 30, 60, and 90 degrees. The frequency range of the regular waves was from 3.0 to 9.8 rad/sec.

(d) Test for measuring a quadratic transfer function of surge motion

In order to experimentally obtain the quadratic transfer function of a moored floating structure, the estimation of the cross bispectrum between waves and responses is required as mentioned in the Chapter 2.2. Therefore, in order to generate irregular waves over long duration, the filtered signals were used, which we obtained by means of passing the white noise signals generated from a noise generator into band pass filters. The rolloff(cutoff characteristics) of the band pass filters was 24db/oct.. Four kinds of irregular waves were generated. The central frequencies f of the band pass filter were 0.4, 0.5, 0.6 and 0.7 Hz. In the case of f equal to 0.7 Hz, the duration time of irregular waves was 90 minutes, and for the other cases it was 45 minutes. The encounter angle of these tests is only head sea.

3.4.2 Numerical calculation

(1) Method

Computation of the first order hydrodynamic forces was made by a program based on the three dimensional potential theory. In the computation the mean wetted surface of the body is approximated by 480 facets. The cpu time consumed to calculate the first order forces was about one hour on the FACOM M180 II AD computer. The steady and slowly varying drift forces were calculated by integrating pressure distributions over the wetted surface. The component due to second order potentials was not taken into account. The cpu time for calculation of drift forces was 10 minutes for the same computer.

(2) Check of numerical accuracy

In order to check the numerical accuracy of drift forces, computed results were compared with the Pinkster's. All of calculations were executed in double precisions. Comparisons between ours and Pinkster's are shown in Fig.3.3. In this figure black circles show the present results and broken and solid lines show Pinkster's results. The legends (1), 2), 3), 4)) denote components of steady drift force in Eqs.(2.16) through (2.19) and "total" means a sum of these components. There are a important points to note in this figure. The present calculations for the component 1) in the horizontal mean drift force are less than Pinkster's results. The other three terms and results of the vertical mean drift force agree very well. The component 1) is the largest and is opposite in sign to the components 2),3) and 4), whose sum is comparable in magnitude with the component 1). Thus, small percentage errors in term I give rise to larger percentage errors in the total drift force. The differences in the component 1) are also certainly due to the difference of the way modelling the waterline.

3.4.3 Hydrodynamic force characteristics of surge motion

(a) Free oscillation test in still water

An example of experimental results is shown in Fig.4. By using this data, a virtual mass and equivalent damping coefficient were obtained as follows:

Let x_n be sequential peak values (amplitudes) of damping curve. And it is assumed that the decaying motion can be represented by:

$$x = X_0 \exp\left[-\frac{N_{11}^e}{2(M_1 + m_{11})}\right] \sin\left(\frac{2\pi t}{T_0} + \Psi\right) \quad (3.36)$$

where T_0 is the natural period, $(M_1 + m_{11})$ the virtual mass, and N_{11}^e the equivalent linearized damping coefficient. Then if we plot $|x_{n+2} - x_{n+1}|$ as a function $|x_{n+1} - x_n|$ and the damping is constant, from Eq.(3.36) we get:

$$|x_{n+2} - x_{n+1}| = \exp\left[-\frac{N_{11}^e}{4(M_1 + m_{11})}\right] |x_{n+1} - x_n| \quad (3.37)$$

Thus by the least square method, the minimum error estimate of the inclination Θ can be obtained. The natural period T_0 is obtained from the mean of zero-upcrossing periods and zero-downcrossing periods. Then the virtual mass and equivalent damping coefficient are given by:

$$M_1 + m_{11} = \frac{T_0^2 C_{11}}{4\pi^2} \quad (3.38)$$

$$N_{11}^e = -\frac{T_0 C_{11} \log(\Theta)}{\pi^2} \quad (3.39)$$

where C_{11} is a restoring force coefficient.

The results obtained in this way are shown in Table 3.2. In order to apply this method, a large number of peak values is required and the motion equation must be linear. If the number of peak values is small, the accuracy of hydrodynamic force coefficients will become poor. So we must study whether the hydrodynamic coefficients obtained from Eqs.(3.38) and (3.39) have a good accuracy.

(b) Forced irregular oscillation test in still water

The forced irregular oscillation tests were carried out in still water by using the surge motion signals (including the slow drift motion) obtained from the motion measurement experiments in waves. This test was done to study the accuracy of the hydrodynamic force coefficients obtained from the free oscillation test. The hydrodynamic force coefficients by this test are given as follows:

Let S_{xF} be a cross-spectrum between the forced surge displacement x and the hydrodynamic reaction force F and S_x be a auto-spectrum of x . Then the hydrodynamic force coefficients can be obtained from the following equation.

$$C_{11} - (M_1 + m_{11}) = \Re\left\{\frac{S_{xF}(\omega)}{S_x(\omega)}\right\} \quad (3.40)$$

$$N_{11}^e = \Im \left\{ \frac{\frac{S_{xy}(\omega)}{S_x(\omega)}}{\omega} \right\} \quad (3.41)$$

This method can be applied only to the case of linear motion equation. If the hydrodynamic forces in the motion equation are nonlinear, note that those coefficients obtained by this method express nothing but equivalent linearized coefficients. Comparison between the surge hydrodynamic coefficients obtained from the free oscillation test and ones from the irregular forced oscillation test is shown in Fig.3.6. The horizontal axis indicates the non-dimensional frequency $\hat{\omega} = \omega \sqrt{\frac{D}{g}}$, where D is the diameter of one column and ω is the surge motion frequency. In this figure white circles show the hydrodynamic coefficients obtained from the irregular forced oscillation test while black circle show those from the free oscillation test. The broken line indicates the numerical results calculated by the three dimensional source distribution method. The damping force coefficient is nondimensionalized by $\rho \nabla \sqrt{\frac{g}{D}}$.

The inertia coefficients obtained from the forced irregular oscillation test are distributed around the numerical values calculated by the three dimensional source distribution method while those from the free oscillation test agree well with the numerical values. The equivalent damping coefficients from the irregular forced oscillation test take negative value in some frequency range and distribute in the wide region from -0.1 to 0.2. Both results from the forced irregular oscillation test and the free oscillation test are in rough agreement. From this figure it is found that the inertia force in low frequencies can roughly be predicted from the three dimensional potential theory and the damping force can be obtained from the free oscillation test in still water. But in general, it is well known that the hydrodynamic forces depend on the magnitude of motion displacement. Thus in order to investigate the motion displacement dependency of hydrodynamic forces, the sinusoidal forced oscillation test was carried out. The motion amplitudes in this test were changed from 3.75 to 15cm, and the motion period was a constant period (17.5 sec., i.e. $\omega=0.0429$). Results are shown in Fig.3.7. The horizontal axis is the Keulegan-Carpenter number (K_c number), which is defined by $2\pi X_0/D$ (where X_0 is the motion amplitude and D is the column diameter). The solid line indicates the results obtained from the free oscillation test, and the broken line shows the results calculated by using the three dimensional source distribution method. From this figure it is seen that the hydrodynamic forces acting on this structure do not depend much on the K_c number against our expectation. However one of the authors and Takaiwa⁷⁾ have conducted the forced and free damping tests for a tanker, a box-shape barge, and a semisubmersible, and they obtained the K_c number dependency of drag coefficients for these structures. According to their results, the drag coefficients appear to be inverse proportional to K_c number in the range of small K_c number. This means that the equivalent damping coefficients do not depend on K_c number in this range of K_c number, but the further researches will be

required to examine this problem.

Within this experiment, inertia force coefficient $(1 + m_{11}/M_1)$ in low frequencies can roughly be estimated at 2.0, and the equivalent damping coefficient (N_{11}^c) is about $4.6 \text{ kg} \cdot \text{sec}/\text{m}$ ($3.56 \text{ ton} \cdot \text{sec}/\text{m}$ in the prototype structure).

3.4.4 Frequency response functions of surge motion

The spectra of irregular waves generated in the experiments are shown in Fig.3.8. And the statistical values are indicated in Table 3.3. The Blackman-Tukey method was used in the spectral analysis. The number of lags was 256 and the Hamming window was used. The number of data taken for the analysis was about 35500 in the case of wave condition 4 and it was about 23000 in the other cases. The sampling interval was 120msec for the analysis and it was 60msec when the data were measured.

In order to get the quadratic transfer functions we need the cross bispectrum estimates as mentioned in Appendix B. The utilization of the Fast Fourier Transform have significant advantage to compute the full components of the cross bispectrum. For present purpose however the full computation is not required, only results on or near the line $\omega_1 = \omega_2$ in bi-frequency plane are needed because our discussion concentrates upon slowly varying forces. Thus, we used the method developed by Dalzell⁸⁾ to estimate the cross bispectrum. The window function used in the computation of cross bispectrum was the Hamming type extended to two dimensions. The coefficients of the window function, i.e. e_1 and e_2 were 0.54 and 0.46 respectively.

For the spectral analysis based on the Blackman-Tukey method the maximum lag number must be less than 1/10 of sampling data. And Dalzell⁸⁾ showed that in order to get a stable cross bispectrum, the maximum lag number must be less than 1/200 or 1/250 of sampling data. Furthermore, as shown by Appendix B, if the lag number of the spectrum analysis is m , one of cross bispectrum analysis becomes $m/2$. In this case we decided that m was 256.

The auto spectra of surge motion are shown in Fig.3.9. The surge response in the case of wave condition 4 is the largest in the four wave conditions and low frequency motions are most dominant in the surge responses. The first order frequency response function, which is obtained from the cross spectra between the surge motion and waves, is shown in Fig.3.10. In the figure, the white circles indicate the experimental results. The solid line shows the theoretical value due to the usual linear motion prediction method which takes into account the viscous damping force obtained from the experiments(see Chapter 3.4.3). The experimental results and the linear theoretical curve are in good agreement.

3.4.5 Characteristics of steady drift force

The steady wave drift forces in wave direction are shown in Figs.3.11 through 3.14. In these figures, χ means a encounter angle to waves and circles indicate

the experimental results, where black circles are for the experimental results with the wave height higher than 7 cm and white circles are for ones with the wave height lower than that. The solid line shows the theoretical curve based on the potential theory and the dotted line shows the modified theoretical curve obtained by taking into account of the viscous drift force (this will be mentioned later) in addition to the potential theory. Fine lines indicate the results obtained from the experiment in irregular waves as follows:

As indicated in the previous section or Appendix B, if the cross bispectrum estimates between waves and second order forces can be directly obtained from the experiment in irregular waves, the frequency response characteristics of drift forces can be estimated with good accuracy. But it is difficult to measure the wave forces including the second order forces when the body is oscillating. Thus we adopted the indirect method instead of the direct measuring method of wave forces. First, we estimated the quadratic transfer function G_2 from the cross bispectrum between the surge motion and waves. Second, we determined the frequency characteristics of the steady drift force in irregular waves by the product between the diagonal components of G_2 and the spring constant. In these figures, the abscissa expresses the non-dimensional wave frequency $\hat{\omega}$, the vertical axis means the mean (steady) drift force coefficients in wave direction, those are normalized by $\frac{1}{2}\rho g\zeta_0^2 L$ (where L is the total length of the floating body and ζ_0 is the incident wave amplitude). And H/D is the ratio between the wave height and the diameter of column. When $H/D < 0.5$, that is, the wave height is less than half of the column diameter, the experimental results agree well with the theoretical line based on the potential theory. But, when H/D becomes larger than 0.5, both results are different considerably. As the cause of the difference, the following physical factors may be considered:

(a) Viscous drift force(surface force):

This occurs from the product of a wave force term, which is in proportion to a squared fluid velocity in the Morison equation, and a wave surface elevation. Chakravarti⁹⁾ and Standing¹⁰⁾ has reported that this force exists.

(b) Steady force due to other viscous drag force:

A vertical viscous drag force changes by angle of pitch motion. And its force produces the horizontal viscous force. Huse¹¹⁾ expressed the horizontal steady viscous force as:

$$F_{vis} = - \langle C_{dZ} v_z | v_z | \xi_{51} \rangle \quad (3.42)$$

where $\langle \cdot \rangle$ denotes time mean value, v_z is a relative vertical velocity between a vertical wave particle velocity and a heaving velocity and ξ_{51} is the pitch motion and C_{dZ} is the vertical drag coefficient.

(c) Steady force due to mass transfer velocity of waves:

Stokes¹²⁾ has shown that the horizontal mean velocity in the direction of wave propagation occurs in the vicinity of wave surface and this phenomenon is caused by the nonlinearity of free surface condition. This velocity is in proportion to the squared wave height. If a steady drag force can be produced by the mass transfer velocity, it may be proportional to the fourth power of wave height.

(d) Drift force due to the second order potentials:

Standing and the others¹⁰⁾ has shown that the second order potential makes no contribution to the horizontal steady force or the steady turning moment. The absence of a steady drift force due to the second order wave can also be explained in physical terms. A steady force would imply the presence of a mean pressure gradient, which would in turn imply a steady acceleration throughout the fluid. This is not possible in the horizontal steady state situation.

In the four factors, we need not consider (d) because the drift force due to the second order potential does not produce a steady force.

Figure 3.15 a) shows the variation of steady drift force coefficient vs. the wave height at the wave frequency ω equal to 4.387 rad/sec (1.16 rad/sec in actual structure, and 0.5254 in the non-dimensional frequency). And Fig. 3.15 b) shows it for each wave frequencies. It is clear from these figures that the steady drift force coefficient linearly increases with an increase of the wave height when H/D is greater than 0.5. This means that the steady drift force is proportional to the third power of wave height when $H/D > 0.5$. Thus the factor (c) is not considered. If the factor (b) is significant, the steady drift force component (4) represented by Eq.(2.19) must also be significant. Because since the first order wave force in the vertical direction includes the force component proportional to the vertical velocity v_y , the component (4), in natural, becomes large when the factor (b) is dominant compared with other factors. Thus, we studied the contribution ratio of each steady drift force components ((1) to (4)) to the total steady drift force by numerical calculations. Figure 3.16 shows the results.

As found from the figure, for this structure, the force components (1) and (2) are dominant compared with other components, that is, the contribution of the force components (3) and (4) to the total force is very small compared with the force components (1) and (2). Accordingly, also the factor (b) is not dominant. Finally the phenomenon, which the steady drift force is proportional to the third power of wave height in some frequency range, is caused by the viscous drift force or surface (drift) force.

Since it is very difficult to strictly evaluate this force, we shall study the force on a simple vertical circular cylinder within the linear wave theory. This investigation is referred to Appendix C. This viscous drift force has the following characteristics.

- (1) The viscous drift force is in proportion to the third power of wave height and it is expressed by the product of the horizontal drag force in the Morison equation and the instantaneous wave surface elevation. And if the drag force can equivalently be linearized, the viscous drift force can also be represented by the second term in the Volterra functional power series.
- (2) The slowly varying viscous drift force increases with increasing the mean wave frequency of two different wave components.
- (3) The viscous drift force does not depend on the draft but the ratio between the wave height and the diameter of the cylinder.

The second result shows that the slowly varying drift force including the viscous drift force can be expressed by the second term of the Volterra functional series. But in order to strictly deal with the viscous drift force, it is necessary to take into account the interaction between viscous and potential flows, furthermore we must consider the problems of diffraction and memory effects in the Morison equation.

For simplicity, we applied the Standing's method to estimate the viscous drift force acting on the structure considered.

Standing⁹⁾ has shown the relation between the steady viscous drift force and the potential drift force on a fixed vertical circular cylinder, resting on the sea-bed and piercing the free surface as follows:

$$R = \left\{ \frac{\frac{H}{D}}{2\pi^3 \frac{D^2}{\lambda} C_d} \right\} \quad (3.43)$$

where D is a diameter of the cylinder, H the wave height, λ the wave length and C_d the drag coefficient.

Figure 3.17 shows the contribution rate of viscous and potential components to the steady drift force. The dotted line indicates a wave breaking limit. White circles are the experimental results and the solid line shows the curve of R equal to 1, i.e., the viscous steady drift force is equal to the potential steady drift force, when $C_d = 1$. It is clear that the viscous steady drift force is larger than the potential one when $H/D > 0.5$. Thus if the ratio of the viscous drift force to the potential one is high, we must take into account the viscous drift force as follows:

$$\bar{F}_d = (1 + R)F_d \quad (3.44)$$

where F_d is the potential steady drift force and \bar{F}_d is the steady drift force corrected by viscous effect, i.e., the steady drift force including both the viscous and potential drift forces.

In the case of experiments in irregular waves, H is replaced by half of the significant wave height and C_d is 0.5. The drag coefficient was obtained from a result of the towing test. \bar{F}_d is shown by the thick dotted line in Fig.3.17. From this figure it is found that the estimate of the steady drift force corrected by viscous effect agrees with the experimental results.

3.4.6 Characteristics of slowly varying drift force

Numerical contours of real and imaginary parts of slowly varying drift force f_{ij} are shown as a function of two variables ω_i and ω_j in Fig.3.18. The variables $\hat{\omega}_i$ and \hat{f}_{ij} are normalized by:

$$\hat{\omega}_i = \omega_i \sqrt{\frac{D}{g}} \quad (3.45)$$

$$\hat{f}_{ij} = \frac{f_{ij}}{\frac{1}{2}\rho g |a_i| |a_j| L} \quad (3.46)$$

where a_i and a_j are amplitudes of two different waves respectively.

It is found from this figure that the real part of \hat{f}_{ij} has the peak in the vicinity of $(\hat{\omega}_i, \hat{\omega}_j) = (0.806, 0.806)$, but it is flat except in the vicinity, and that the imaginary part is also flat along the line $\hat{\omega}_i = \hat{\omega}_j$. This result may infer that the Newman approximation can be applied to this model.

Comparison between the numerical and experimental results with respect to the slowly varying drift force is shown in Fig.3.19. The left side indicates the amplitude of quadratic transfer function of slowly varying drift force and the right side does the phase of it. The thin dotted lines are the experimental results in irregular waves (those results are obtained from the cross bispectrum analysis), the solid line is the numerical results based on the potential theory, the dash-dotted line obtained from applying the Newman approximation to the numerical results, and the broken line the results obtained from applying the Newman approximation to the numerical values corrected by the viscous effect; i.e. the values estimated by Eq.(3.44). And $\Delta\omega$ indicates the difference of two different wave frequencies and the horizontal axis is the mean frequency of them.

Although the slowly varying drift force may directly be obtained from the experiment, we indirectly obtained the force in the following way.

Using the quadratic transfer function of surge motion, G_2 (which is obtained from cross bispectral analysis of the experimental data) and the transfer function of surge motion to the external force, H_L (which is obtained from the free oscillation test in still water), the quadratic transfer function of the slowly varying drift force G_2^f can indirectly be obtained by the following relation:

$$G_2^f(\omega_1, -\omega_2) = \frac{G_2(\omega_1, -\omega_2)}{H_L(\omega_1 - \omega_2)} \quad (3.47)$$

where

$$H_L(\omega) = \frac{1}{C_{11} - (M + m_{11})\omega^2 + iN_{11}^e\omega} \quad (3.48)$$

and we assume that the hydrodynamic force coefficients of Eq.(3.48) do not change in waves.

From this figure the numerical value based on the potential theory is much lower than the experimental results, but the former has the same tendency as the latter. Comparisons of the solid and dash-dotted lines reveal that the Newman approximation can be applied in this case. And the broken line, i.e. the numerical results corrected by taking into account the viscous effect, agrees with the experimental results. This means that in order to estimate the slow drift motion of the floating structure supported by many legs with small diameter, we should take into account not only the potential drift force but also the viscous one.

3.4.7 Variation of hydrodynamic force coefficients of slow drift motion in waves

In the section 2.1, we state that the damping coefficient of slow drift motion in waves is different from one in still water. In this section we shall investigate if such phenomenon occurs in the following way.

First, let G_2 , G_2^f and H_L be the quadratic transfer function of surge motion, the quadratic transfer function of slowly varying drift force and the transfer function of surge motion to external force, respectively. Let them hold the relationship of Eq.(3.47). And we shall introduce the transfer function of slow drift motion to instantaneous wave power, $\Xi(\omega)$, given by :

$$\begin{aligned}\Xi(\omega) &= \frac{S_{x\zeta^2}}{S_{\zeta^2}(\omega)} \\ &= \frac{\int_{\omega'} S_{\zeta}(\omega - \omega') S_{\zeta}(\omega') G_2^*(\omega - \omega', \omega') d\omega'}{\int_{\omega'} S_{\zeta}(\omega - \omega') S_{\zeta}(\omega') d\omega'}\end{aligned}\quad (3.49)$$

where $S_{x\zeta^2}$ is the cross spectrum between the surge motion x and instantaneous wave power ζ^2 , and S_{ζ^2} the auto spectrum of ζ^2 . Then from Eqs.(3.23) and (3.47), the following relation is satisfied:

$$\Xi^*(\omega) = H_L(\omega) W_2(\omega) \quad (3.50)$$

Thus, if the Newman approximation can be applied, the non-dimensional transfer function of surge motion to external force, \bar{H}_L can be obtained by:

$$\bar{H}_L(\omega) = \frac{\Xi^*(\omega)}{\Xi(0)} \quad (3.51)$$

Comparison between \bar{H}_L obtained from Eq.(3.51) and $\tilde{H}_L (= C_{11} \cdot H_L)$ obtained from Eq.(3.48) is shown in Fig.3.20. In the figure the thin lines are the results of \bar{H}_L and the solid line is the result of \tilde{H}_L , where the value $\Xi(0) (= H_L(0) \cdot W_2(0))$

in Eq.(3.51) is estimated from (3.26) as follows:

$$\Xi(0) = \frac{\overline{F}^{(2)}}{C_{11}\sigma_{\zeta}^2} \quad (3.52)$$

From this figure, \overline{H}_L is in good agreement with \tilde{H}_L in case of wave condition 1, but in other cases, the peak frequency of \overline{H}_L moves towards the low frequency side and the peak value becomes small when the peak frequency of wave spectrum becomes high, as compared with \tilde{H}_L . Namely, this means that when the peak frequency of wave spectrum becomes high, the damping coefficient of slow drift motion in waves becomes bigger than one in still water. In order to examine an increase rate in the damping coefficient, we got the hydrodynamic coefficients by means of the least square method from Eq.(3.52), under the assumption that \overline{H}_L is equivalent to Eq.(3.48). These results are shown in Table 3.5. Obviously, the phenomenon that the damping force in waves becomes larger than one in still water occurs. The amount is 1.6 ~ 1.7 times the damping force in still water. Furthermore the virtual mass in waves decreases 10 % of one in still water.

3.4.8 Time domain simulation

(1) Surge motion equation in time domain and its solution

If the added mass and the damping forces of slow drift motion in still water do not change in waves and the coupling terms are neglected, a surge motion equation of the floating body moored by linear springs may be represented in time domain as follows:

$$\begin{aligned} (M_1 + m_{11}(\infty))\ddot{X}_1 + \int_{-\infty}^t K_{11}(t - \tau)\dot{X}_1 d\tau + a_{11}(\dot{X}_1, \dot{\zeta}; t) + C_{11}X_1 \\ = F^{(1)}(t) + F^{(2)}(t) \end{aligned} \quad (3.53)$$

where

- M_1 ; mass
- $m_{11}(\infty)$; added mass at $\omega = \infty$
- a_{11} ; viscous damping force
- C_{11} ; restoring force coefficient
- K_{11} ; memory effect function
- $F^{(1)}$; first order force
- $F^{(2)}$; second order force

Moving all terms in Eq.(3.53) except for inertia terms to the right hand side, Eq.(3.53) becomes equivalent to the Newton equation as:

$$M_1\ddot{X}_1 = -m_{11}(\infty)\ddot{X}_1 - \int_{-\infty}^t K_{11}(t - \tau)\dot{X}_1 d\tau - a_{11}(\dot{X}_1, \dot{\zeta}; t) - C_{11}X_1$$

$$+F^{(1)}(t) + F^{(2)}(t) \quad (3.54)$$

Then we can numerically solve the above equation in time domain if the viscous damping force is known. In order to solve Eq.(3.54) in time domain, we used the Newmark- β method¹⁴⁾. According to the Newmark- β method, when a surge motion at a time t_n is expressed by X_1^n , X_1^{n+1} at $t_{n+1} = t_n + \Delta t$ can be represented as follows:

$$X_1^{n+1} = X_1^n + \Delta t \dot{X}_1^n + \frac{\Delta t^2}{2} \ddot{X}_1^n + \beta \Delta t^2 (\ddot{X}_1^{n+1} - \ddot{X}_1^n) \quad (3.55)$$

$$\dot{X}_1^{n+1} = \dot{X}_1^n + \frac{\Delta}{2} (\ddot{X}_1^{n+1} + \ddot{X}_1^n) \quad (3.56)$$

After iterations, the motion equation, that is, Eq.(3.54) can be solved in time domain, where we use 1/4 as a value of β . When this value is used, it is mathematically proven that the solution is absolutely stable.

The judgement of convergence was conducted under the following condition:

$$\left| \frac{\ddot{X}_{1,m}^{n+1} - \ddot{X}_{1,m}^n}{\ddot{X}_{1,m}^{n+1}} \right| \leq \frac{1}{100} \quad (3.57)$$

where the subscript m denotes the iteration number.

(2) Hydrodynamic force in time domain

From the Fourier transform to the first two terms in the left hand side of Eq.(3.53), the following relations are given;

$$m_{11}(\omega) = m_{11}(\infty) - \frac{1}{\omega} \int_0^\infty K_{11}(t) \sin \omega t dt \quad (3.58)$$

$$N_{11}^{(1)}(\omega) = \int_0^\infty K_{11}(t) \cos \omega t dt \quad (3.59)$$

where

$m_{11}(\omega)$; added mass in frequency domain

$N_{11}^{(1)}(\omega)$; radiation wave damping in frequency domain

Then if the added mass and the radiation wave damping force over infinite range are given, the hydrodynamic forces in time domain, i.e. $m_{11}(\infty)$ and $K_{11}(t)$, can be obtained from the relations (3.58) and (3.59). But this procedure is not easy, because it is impossible to get the frequency-domain hydrodynamic force numerically over infinite range. Thus we extrapolate $N_{11}^{(1)}(\omega)$, which is obtained in some frequency range, by using the spline function, get the frequency

point ω_0 that the extrapolated value becomes zero, and calculate the following integral over $\omega_0 \geq \omega \geq 0$.

$$K_{11}(t) = \frac{2}{\pi} \int_0^{\omega_0} N_{11}^{(1)}(\omega) \cos \omega t d\omega \quad (3.60)$$

We will check the accuracy of the above numerical approximation later.

(3) Viscous hydrodynamic force

In order to get the viscous hydrodynamic damping forces, one divides wetted surfaces of a floating body into several blocks, and obtains the viscous damping force from integrating the viscous drag acting on the centre of projection area of all blocks: That is

$$a_{11} = N_{11}^{(2)} \dot{X}_1 | \dot{X}_1 | \quad (3.61)$$

$$N_{11}^{(2)} = \frac{1}{2} \rho \iint_S n_1 C_d dS \quad (3.62)$$

In this paper, for simplicity, the viscous drag force was determined by the following equivalent linearized form:

$$a_{11} = N_{11}^e \dot{X}_1 \quad (3.63)$$

where the experimental value shown in the section 3.4.3 was used as the value of N_{11}^e in this case.

(4) Wave force

(a) First order force

According to the linear system theory in the field of communication theory, the first order wave excitation acting on a floating body can be represented as;

$$F^{(1)}(t) = \int g_1^f(t) \zeta(t - \tau) d\tau \quad (3.64)$$

where g_1^f is a impulse response function of first order wave excitation, and its Fourier transform, i.e. frequency response function, becomes as:

$$g_1^f(\tau) = \frac{1}{2\pi} \int G_1^f(\omega) \exp(i\omega t) d\omega \quad (3.65)$$

If the wave spectrum shape $U(\omega)$ is known, Rice has shown that the first order wave excitation can be represented by the following stochastic integral form:

$$F^{(1)}(t) = \int |G_1^f(\omega)| \cos(\omega t + \mu(\omega) - \arg(G_1^f(\omega))) \sqrt{2U(\omega)} d\omega \quad (3.66)$$

Note that the Rice's representation does not depend on the initial value, i.e. it is a stochastic integral representation, and it is not a physical causal system.

(b) Steady and slowly varying drift forces

Using the system function w_2 defined in Eq.(3.18), the slowly varying drift force including the steady drift force can be represented as:

$$F^{(2)}(t) = \int w_2(\tau) \zeta^2(t - \tau) d\tau \quad (3.67)$$

where

$$w_2(\tau) = \frac{1}{2\pi} \int W_2(\omega) \exp(i\omega\tau) d\omega \quad (3.68)$$

and

$$W_2(\omega) = \frac{1}{\sigma_\zeta^2} \int G_2^f(\omega - \omega', \omega') S_\zeta(\omega') d\omega' \quad (3.69)$$

If $G_2^f(\omega_i, \omega_j)$ is smooth enough for ω_i and ω_j and $\frac{\partial G_2^f}{\partial \omega_i}$ and $\frac{\partial G_2^f}{\partial \omega_j}$ are small, we can generate the slowly varying drift force by passing $\frac{F^{(2)}}{\sigma_\zeta^2} \cdot \zeta^2(t)$ into a low pass filter, as shown in the section 3.2.

(5) Comparison between simulation results and experimental ones

Before doing the simulation we investigated that the assumption (3.60) can be applied. Takagi and Saito¹⁵⁾ has shown theoretically an asymptotic behaviour of the memory effect functions for a half submerged sphere. Comparisons between their results and the calculated results due to Eq.(3.60) are shown in Fig.3.21. It is found from this figure that both results are in agreement although a slight deformation is observed to the calculated memory effect function. It is considered from practical point of view that the present calculation method is accurate enough to get memory effect functions since in general radiation damping forces exponentially decrease with increasing wave frequency. However we should note that the added mass $m_{11}(\infty)$ is slightly modified by the truncation effect.(see e.g. Fig.3.22). In this paper calculations were carried out until the frequency range such that a stable added mass, $m_{11}(\infty)$ is given.

Comparisons between simulation results due to Eq.(3.53) and experimental results of slow drift surge motion for each wave conditions are shown in Figs.3.23 and 3.24, and the surge motion spectra of each results are indicated in Fig.3.25. The slowly varying drift forces are simulated by using both Eqs.(3.18) and (3.27). As an amplitude of $\vartheta(\omega)$, which expresses the frequency characteristics of a low pass filter in Eq.(3.27), a squared cosine type such that $\vartheta(\omega) = 1$ for $\omega = 0$ and $\vartheta(\omega) = 0$ for ω equal to the peak frequency of wave spectrum is used. A time interval for simulation is 60msec. From this figure it is found that both results are in good agreement in the case that the first order motion is dominant, but that the simulation results become larger than the experimental results when the slow drift motion is dominant. It is considered that this is caused by a wave drift damping force.

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Chapter 4

Stochastic analysis of second order responses

This section develops a theory of two probabilistic subjects associated with obtaining the second order response of a moored floating structure in the horizontal plane. The first method utilized to obtain the total second order response p.d.f. assumes neither a weakly nonlinear response nor a pure quadratic response. This theory is based on the "approximate theory" of continuous distribution in mathematical statistics where the p.d.f. of the total second order response can be represented by the Laguerre expansion which express the first term by a Gamma p.d.f. This is similar to the Vinje's method which is comparable to the Gram-Charlier expansion which expresses the first term by a Gaussian p.d.f., although the Gram-Charlier expansion does not uniformly converge and negative probabilities may occur. The use of the Laguerre expansion/Gamma p.d.f. method to obtain the total second order response p.d.f. can be applied to solve the above problems, furthermore it can also treat the case of equal double eigenvalues that the Naess' method cannot. The second method utilized obtains the highest mean amplitude of the total second order response of a moored floating structure. By introducing an assumption that a response and its time derivative processes are mutually independent, it is shown that the p.d.f. of the positive maxima or the negative minima can be expressed by the derivative of the p.d.f. of the instantaneous response.

As a basic study, the applicability of the present method is first discussed by comparisons between the Naess' exact p.d.f. solution for pure second order responses of moored floating semi-circular and rectangular 2- D structures. Next, the statistical interferences of the linear and quadratic responses on the p.d.f. and the $1/n$ th highest mean amplitude are investigated by changing the damping and restoration coefficients of the response system. Finally we investigate the practicability of the present method through comparisons between the

measured results and the estimates obtained from the present method.

4.1 Probabilistic Approach to The Total Second Order Response of a Moored Floating Structure

4.1.1 Instantaneous p.d.f.

(1) Exact Theory

The total second order response of a moored floating structure that is being subjected to a Gaussian random excitation at some fixed time may be expressed as:

$$X(t) = X^{(1)} + X^{(2)} \quad (4.1)$$

where the linear term is given by:

$$X^{(1)} = \int_{\tau} g_1(\tau) \zeta(t - \tau) d\tau \quad (4.2)$$

and the nonlinear second order term as:

$$X^{(2)} = \int_{\tau_1} \int_{\tau_2} g_2(\tau_1, \tau_2) \zeta(t - \tau_1) \zeta(t - \tau_2) d\tau_1 d\tau_2 \quad (4.3)$$

In equations (4.2) and (4.3), $\zeta(t)$ denotes the surface elevation which is a stationary Gaussian random variable with a zero mean. The kernel g_1 is a linear impulse response function. The kernel g_2 is analogous to the linear impulse response function and is called the quadratic impulse response function (see 3.1). And we assume that they are continuous and absolutely integrable, then they possess a Fourier transform as shown previously (Eq.(3.7)).

In order to represent the quadratic process $X^{(2)}$ by a sum of random variables, yielding the same probability distribution, the Kac & Siegert theory (K-S method) is used. This leads to the following representation:

$$X(t) = \sum_j c_j W_j(t) + \sum_j \lambda_j W_j^2(t) \quad (4.4)$$

where W_j is a set of independent Gaussian random variables of zero mean value and unit variance. The λ_j are eigenvalues which satisfy:

$$\int_{-\infty}^{\infty} K(\omega_1, \omega_2) \Psi_j(\omega_2) d\omega_2 = \lambda_j \Psi_j(\omega_1) \quad (4.5)$$

The parameters c_j , which represent the linear response, can be determined by:

$$c_j = \int_{-\infty}^{\infty} G_1(\omega) \sqrt{S_\zeta(\omega)} \Psi_j^*(\omega) d\omega \quad (4.6)$$

where * indicates a complex conjugate and S_ζ is a two-sided wave spectrum. In equation (4.5) is a set of orthogonal eigenfunctions which satisfies:

$$\int_{-\infty}^{\infty} \Psi_j(\omega) \Psi_k^*(\omega) d\omega = \begin{cases} 1 & , j = k \\ 0 & , j \neq k \end{cases} \quad (4.7)$$

and kernel function $K(\omega_1, \omega_2)$ is a Hermite kernel defined by:

$$K(\omega_1, \omega_2) = \sqrt{S_\zeta(\omega_1) S_\zeta(\omega_2)} G_2(\omega_1, \omega_2) \quad (4.8)$$

When the eigenvalues λ_j and the parameters c_j are known, the p.d.f. is given by:

$$p_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ixs) \phi_X(s) ds \quad (4.9)$$

where the characteristic function is

$$\phi_X(s) = \prod_{j=1}^{\infty} \frac{1}{\sqrt{1 - 2i\lambda_j s}} \exp\left[-\frac{c_j^2 s^2}{\sqrt{2(1 - 2i\lambda_j s)}}\right] \quad (4.10)$$

The mean, the variance and the higher order cumulants are given by:

$$\begin{aligned} k_1 &= \bar{X} = E[X(t)] = \sum \lambda_j \\ k_2 &= \sigma_X^2 = \sum c_j^2 + 2 \sum \lambda_j^2 = \sigma_1^2 + \sigma_2^2 \\ k_m &= \sum 2^{m-1} (m-1)! \lambda_j^m + \sum m! \lambda_j^{m-2} c_j^2 \quad \text{for } m \geq 3 \end{aligned} \quad (4.11)$$

Kac and Siegert¹⁾ and Neal²⁾ concluded that the p.d.f. expressed in Eq.(4.9) cannot be determined in a closed form and therefore must be computed numerically. Although this is true in most cases, it can be written in a closed form in some special cases which will be discussed next.

(2) NAESS' APPROACH

Naess^{3),4),5)} introduced a slow drift approximation such that $G_2(\omega_1, \omega_2) = 0$ when $\omega_1 \cdot \omega_2 > 0$. This indicates that the high frequency component which corresponds to sum of ω_1 and ω_2 is negligible. This is a physically acceptable fact, and it is a convenient approximation for our purpose. Naess determined

that the Eq.(4.5) eigenvalue problem generated a set of double eigenvalues as follows:

$$K(\omega_1, \omega_2) = 0 \quad \text{for } \omega_1 \cdot \omega_2 < 0$$

$$\int_0^\infty K(\omega_1, \omega_2) \tilde{\Psi}_j(\omega_2) d\omega_2 = \tilde{\lambda}_j \tilde{\Psi}_j(\omega_1) \quad \text{for } \omega_1 \geq 0 \quad (4.12)$$

$$\lambda_{2j-1} = \lambda_{2j} = \tilde{\lambda}_j$$

where

$$\Psi_{2j-1}(\omega) = \begin{cases} \frac{1}{\sqrt{2}} \tilde{\Psi}_j(\omega) & , \omega \geq 0 \\ \frac{1}{\sqrt{2}} \tilde{\Psi}_j^*(\omega) & , \omega < 0 \end{cases} \quad (4.13)$$

$$\Psi_{2j}(\omega) = \begin{cases} -\frac{i}{\sqrt{2}} \tilde{\Psi}_j(\omega) & , \omega > 0 \\ 0 & , \omega = 0 \\ \frac{i}{\sqrt{2}} \tilde{\Psi}_j^*(\omega) & , \omega < 0 \end{cases}$$

and also that the linear response is negligibly small when compared to the second order response, i.e., $c_j \equiv 0$. The p.d.f. of the pure second order response can be shown in the closed form as follows:

$$p_X(x) = \begin{cases} \sum \frac{l_j}{2\lambda_j} \exp(-\frac{x}{2\lambda_j}) & , x \geq 0 \\ \sum \frac{l_j}{2|\lambda_j|} \exp(\frac{x}{2|\lambda_j|}) & , x < 0 \end{cases} \quad (4.14)$$

where

$$l_j = \prod_{\substack{k=1 \\ k \neq j}}^N \frac{1}{(1 - \frac{\tilde{\lambda}_k}{\lambda_j})} \quad (4.15)$$

and the set of eigenvalues g are divided into two groups, $\lambda_j, j = 1, \dots, M$, for $\lambda_j > 0$ and $\lambda_j, j = M + 1, \dots, N$, for $\lambda_j < 0$. The above results are then valid unless equal double eigenvalues exist.

(3) APPROXIMATE THEORY

(i) Gram-Charlier expansion method

The authors⁶⁾ showed that if the nonlinear response considered here is weakly nonlinear the instantaneous p.d.f. can be represented by the Gram-Charlier expansion. The expansion is the Hermite expansion, the first approximation of which is the Gaussian p.d.f.. We shall indicate their method in brief.

If the eigenvalues λ_j are very small compared with c_j , X may approach Gaussian. So we replace $X - E[X]$ by Z and introduce the error function $p_\epsilon(z)$ defined by

$$p_\epsilon(z) = p_X(z) - N(0, \sigma_X^2) \quad (4.16)$$

where $N(0, \sigma_X^2)$ is the zero mean Gauss p.d.f. with variance equal to σ_X^2 . If p_ϵ can be represented by a family of orthogonal functions with weighting function $\{w(z)h_n(z)\}$, it can be expanded in the following form:

$$p_\epsilon(z) = \sum_{n=1}^{\infty} \alpha_n h_n(z) w(z) \quad (4.17)$$

where

$$\alpha_n = \int_{-\infty}^{\infty} h_n(z) p_\epsilon(z) dz \quad (4.18)$$

If $w(z)$ is the Gaussian p.d.f., it is well-known that $h_n(z)$ are given by the Hermite polynomials. From the properties of the Hermite polynomials the p.d.f. can be approximated by the Gram-Charlier expansion:

$$p_X(x) \simeq \frac{1}{2\pi\sigma_X} \left[1 + \sum_{n=3}^{\infty} \frac{\bar{b}_n}{n!\sigma_X^n} H_n \left\{ \frac{x - \bar{X}}{\sigma_X} \right\} \exp\left(-\frac{x - \bar{X}}{2\sigma_X^2}\right) \right] \quad (4.19)$$

where H_n are the Hermite polynomials and \bar{b}_n represent the higher moments defined by

$$b_n = E[(x - \bar{X})^n] \quad \text{for } n \geq 3 \quad (4.20)$$

And the moments functions can be obtained from frequency domain integrals of transfer functions and wave spectrum as shown in Appendix F.

This method has a significant advantage in the point of obtaining the approximate solution from numerical integral procedures. However, we should note that the Gram-Charlier expansion does not always converge uniformly and that the negative probabilities occur if the expansion is truncated at finite order. The occurrence of negative probabilities is physically inconsistent.

Edgeworth⁷⁾ investigated the convergence of the Gram-Charlier series and he has shown that if only a few terms are computed, the best grouping of terms in Eq.(4.19) is not that associated with taking terms in their natural order (i.e. 0, 3, 4, 5, ...). And he proposed regrouped series. The grouping is

0
0,3 1st approximation
0,3,4,6 2nd
0,3,4,6,5,7,9 .. 3rd

This list implies that if the 0 and 3 terms are used as the first approximation, the addition of terms 4 and 6 gives the next order approximation, and so forth. This regrouped series is called "*Edgeworth series*". The Gram-Charlier series up to third order is equal to the Edgeworth series.