(ii) Asymptotic solution method

Naess[4)] found an exact series form solution for the instantaneous p.d.f. of total second order response. His argument is as follows:

From equation (4.10) and the slow drift approximation, the c.f., $\phi_X(s)$ is given by

$$\phi_X(s) = \prod_{j=1}^{N} \phi_j(s) \tag{4.21}$$

$$\phi_j(s) = \frac{1}{1-2i\tilde{\lambda}_j s}\exp[-\frac{(c_{2j-1}^2+c_{2j}^2)}{2(1-2i\tilde{\lambda}_j s)}] \tag{4.22}$$

It is seen that $\phi_X(s)$ has isolated essential singularities at $s = -\frac{i}{2\tilde{\lambda}_j}$. Rewriting $\phi_j(s)$ as

$$\phi_j(s) = \frac{i}{2\tilde{\lambda}_j(s-s_j)}\exp[-\frac{ib_j s^2}{s-s_j}] \tag{4.23}$$

where $b_j = (c_{2j-1}^2 + c_{2j}^2)4\tilde{\lambda}_j$ and $s_j = -\frac{i}{2\tilde{\lambda}_j}$, it can be shown that

$$\begin{aligned} \phi_X(s)\exp(-ixs) &= \frac{i\tilde{\phi}_j(s)}{2\tilde{\lambda}_j(s-s_j)}\exp\{-i(s_j x + 2b_j s_j)\} \\ &\quad \times \exp\{-i[(b_j+x)(s-s_j)+\frac{b_j^2 s^2}{s-s_j}]\} \end{aligned} \tag{4.24}$$

where the function $\tilde{\phi}_j = \frac{\phi_X(s)}{\phi_j(s)}$. Hence $\tilde{\phi}_j(s)$ is analytic in a neighborhood of s_j, which implies that

$$\tilde{\phi}_j(s) = \sum_{n=0}^{\infty} a_n^{(j)}(s-s_j)^n \quad \text{for } |s| < e_j \ (e_j \text{ are any constants}) \tag{4.25}$$

The p.d.f. can be obtained from integrating (4.24) from $-\infty$ to ∞ with respect to s. Invoking the residue theorem, consequently we get:

$$p_X(x) = \begin{cases} \sum \frac{l_j}{2\tilde{\lambda}_j}\exp(-\frac{x}{2\tilde{\lambda}_j} - \frac{b_j}{\tilde{\lambda}_j})Q_j(x) & x \geq 0 \\ \\ \sum \frac{l_j}{2|\tilde{\lambda}_j|}\exp(\frac{x}{2|\tilde{\lambda}_j|} - \frac{b_j}{\tilde{\lambda}_j})Q_j(x) & x < 0 \end{cases} \tag{4.26}$$

where the function $Q_j(x)$ are defined by

$$Q_j(x) = \sum_{m=0}^{\infty}(\frac{i}{2|\tilde{\lambda}_j|})^m a_m^{(j)}\,(\frac{b_j}{b_j+x})^{m/2} I_m(\frac{\sqrt{b_j(b_j+x)}}{|\tilde{\lambda}_j|}) \tag{4.27}$$

and $I_m(x)$ denotes the modified Bessel function of integer order m. The expansion coefficients $a_m^{(j)}$ can be derived from the Taylor expansion of the function

$$\tilde{\phi}_j(s) = \prod_{\substack{k=1 \\ k\neq j}}^{\infty} \phi_k(s)$$

around $s = s_j$. When $b_j = 0, j = 1, \cdots, N$, i.e. when the first order response is neglected, it is easily seen that since $I_0(0) = 1$ and

$$a_0^{(j)} = \prod_{\substack{k=1 \\ k\neq j}}^{\infty} \phi_k(s_j)$$

,equation (4.26) reduces to equation (4.14).

Since it is very difficult to numerically evaluate equation (4.26), Naess obtained the aymptotic solution for $x \to \infty$ from (4.26) when $\tilde{\lambda}_1$ is dominant compared with the other eigenvalues, i.e. when the following approximation is adopted.

$$\tilde{\phi}_1(s) \simeq \tilde{\phi}_1(s_1) = a_0^{(1)} = \prod_{k=2}^{N} \phi_k(s_1) \tag{4.28}$$

From Eq.(4.22) it is found that

$$a_0^{(1)} = \prod_{j=2}^{N} \frac{1}{(1 - \frac{\tilde{\lambda}_j}{\tilde{\lambda}_1})} \exp[\frac{\tilde{\lambda}_j b_j}{\tilde{\lambda}_1(\tilde{\lambda}_1 - \tilde{\lambda}_j)}] \tag{4.29}$$

Using the following asymptotic relation:

$$I_0(x) \sim \frac{1}{\sqrt{2\pi x}} \exp(x) \quad \text{as } x \to \infty \tag{4.30}$$

it can now be shown that

$$p_X(x) \sim \frac{a_0^{(1)}}{2\sqrt{2\pi\tilde{\lambda}_1}} \left(\frac{\sqrt{b_1(b_1 + x)}}{\tilde{\lambda}_1}\right)^{-1/2} \exp[-\frac{(\sqrt{b_1 + x} + \sqrt{b_1})^2}{2\tilde{\lambda}_1}] \quad \text{as } x \to \infty \tag{4.31}$$

This implies that p_X behaves like $O(\exp(-x))$ for $x \to \infty$ when $\tilde{\lambda}_1 \gg \tilde{\lambda}_j$.

Vinje[8)] also found the same expression as (4.31). But his result is in error as noted by Naess.

(iii) New approximate theory

An alternative approach to Naess' exact solution will now be developed. If the number of the eigenvalues are finite, then from Eq.(4.4) the total response $X(t)$ may be decomposed into the following form:

$$X(t) = Z_1 + Z_2 \tag{4.32}$$

where

$$Z_1 = \sum_{j=1}^{M}(c_j W_j + \lambda_j W_j^2) \tag{4.33}$$

$$Z_2 = \sum_{j=M+1}^{N}(c_j W_j + \lambda_j W_j^2) \tag{4.34}$$

It can be mathematically proven that Z_1 and Z_2 are mutually independent in a statistical sense(e.g. Papoulis[10)]). If the time is fixed and $c_j \equiv 0$, Z_1 becomes a random variable which is always positive while Z_2 is always negative. In this case it can be proven from the approximate theory of continuous random distribution in mathematical statistics that the p.d.f. of Z_1 and $-Z_2$ can be expanded to a series of the generalized Laguerre polynomials[11)]. The first term of the series is the two parameter Gamma p.d.f.. For example, if $Y = \lambda_1 W_1^2 + \cdots + \lambda_n W_n^2$ and $\lambda_j > 0 (i = 1, \cdots, n)$, then the p.d.f. of Y can be expanded by the following series with uniform convergence:

$$p_Y(x) = p_\gamma(x, 2\theta; \frac{\nu}{2})[1 + \sum_{k=1}^{\infty} B_k L_k^{(\frac{\nu}{2}-1)}(\frac{x}{2\theta})] \tag{4.35}$$

where p_γ is the Gamma p.d.f. with two parameters θ and ν, $L_k^{(\frac{\nu}{2})}$ is the generalized Laguerre polynomials, and B_k represents the coefficients determined from the orthogonal property of the Laguerre polynomials. Since the parameters θ and ν are unknown, they can be determined by eliminating B_1 and B_2. Then p_γ becomes a second order approximation for p_Y, and the first and second order moments of p_Y agree with those of p_γ. The same approximation can be also applied in the case of $c_j \neq 0$ by transforming Z_1 in Eq.(4.33) into the following form:

$$Y_1 = Z_1 + \sum \frac{c_j}{4\lambda_j} = \sum \lambda_j V_j^2$$

$$V_j = W_j + \frac{c_j}{2\lambda_j} \tag{4.36}$$

Eq.(4.36) is the same quadratic form of Gaussian random variables as the case for $c_j = 0$, except that $E[V_j(t)] = \frac{c_j}{2\lambda_j} \neq 0$. Since the p.d.f. of $V_j^2(t)$ becomes a non-central χ^2 p.d.f. and V_j are mutually independent, the p.d.f. of Y can be represented by a series form of the non-central χ^2 p.d.f. Using the fact that a non-central χ^2 p.d.f. can be expanded by the generalized Laguerre polynomials, the p.d.f. of Y can also be represented by a series form using a Gamma p.d.f. and a generalized Laguerre polynomials like Eq.(4.35). There is, however, statistical interference between the linear and the quadratic responses at the higher order moments greater than third order(e.g. Eq.(4.11)), thus it is insufficient to adequately describe this statistical interference approximation by using only the leading two terms. Therefore we must extend the two term approximation to at least a three term approximation, i.e. approximate the response by means of a Gamma p.d.f. is defined with three parameters$(\theta, \nu, and \delta)$ in the following form:

$$p_\gamma(x, \delta, 2\theta; \frac{\nu}{2}) = \frac{1}{(2\theta)^{\nu/2}\Gamma(\nu/2)}(x-\delta)^{\nu/2-1}\exp(-\frac{x-\delta}{2\theta})\mho(x-\delta) \quad (4.37)$$

where $\mho(x-\delta)$ is the step function defined as:

$$\mho(x-\delta) = \begin{cases} 1 & x \geq \delta \\ 0 & x < \delta \end{cases} \quad (4.38)$$

θ is the generating number of Gamma p.d.f., and ν the degrees of freedom.
The corresponding c.f. becomes:

$$\phi_\gamma(u, \delta, 2\theta; \nu/2) = \frac{1}{(1-2i\theta u)^{\nu/2}}\exp(i\delta u) \quad (4.39)$$

Taking the difference between the cumulant-generating function of Z_1 and that of a random variable which yields a three parameter (θ, ν, δ) Gamma p.d.f. we obtain,

$$\begin{aligned}\Delta \quad &\equiv \log\phi_{Z_1} - \log\phi_\gamma \\ &= -\frac{1}{2}\sum_{j=1}^{M}\log(1-2i\lambda_j u) + \frac{\nu_1}{2}\log(1-2i\theta_1 u) - \sum_{j=1}^{M}\frac{c_j^2 u^2}{2(1-2i\lambda_j u)} \\ &-i\delta_1 u \end{aligned} \quad (4.40)$$

Substituting $iu = \frac{\xi_1}{1+2\theta_1\xi_1}$ into equation (4.40) we get:

$$\begin{aligned}\Delta \quad &\equiv -\frac{1}{2}\sum_{j=1}^{M}\log[1-2(\lambda_j-\theta_1)\xi_1] + \frac{M-\nu_1}{2}\log(1+2\theta_1\xi_1) \\ &+\sum_{j=1}^{M}\frac{c_j^2}{2}[\frac{\xi_1^2}{\{1-2(\lambda_j-\theta_1)\xi_1\}(1+2\theta_1\xi_1)}] - \frac{\delta_1\xi_1}{(1+2\theta_1\xi_1)} \end{aligned} \quad (4.41)$$

If θ_1 is taken such as $2\theta_1 > max\lambda_j (j = 1, \cdots, M)$, $|2(\lambda_j - \theta_1)\xi_1)| \leq |2\theta_1\xi_1| \leq 1$ for all ξ_1. Thus Δ can be expanded into a uniform convergence power series. Consequently the expansion form of Δ is given by:

$$\Delta \quad = (\sum\lambda_j - \nu_1\theta_1 - \delta_1)\xi_1 + (\sum\lambda_j^2 - 2\sum\lambda_j\theta_1 + \nu_1\theta_1^2 + \sum\frac{c_j^2}{2} + 2\theta_1\delta_1)\xi_1^2$$

$$+[\frac{3}{4}(\sum\lambda_j^3-3\sum\lambda_j^2\theta_1+3\sum\lambda_j\theta_1^2-\nu_1\theta_1^3)-4\theta_1^2\delta_1+\sum\lambda_jc_j^2$$

$$-2\sum c_j^2\theta_1]\xi_1^3+O(\{2\theta_1\xi_1\}^4) \tag{4.42}$$

The first, second and third terms of the right hand side of equation (4.42) may be eliminated if the unknown variables θ_1, ν_1 and δ_1 are determined as follows:

$$\begin{aligned}\theta_1 &= \frac{4\sum\lambda_j^3+3\sum\lambda_jc_j^2}{4\sum\lambda_j^2+2\sum c_j^2}\\ \delta_1 &= \sum\lambda_j-\frac{(2\sum\lambda_j^2+\sum c_j^2)^2}{4\sum\lambda_j^3+3\sum\lambda_jc_j^2}\\ \nu_1 &= \frac{2(2\sum\lambda_j^2+\sum c_j^2)^3}{(4\sum\lambda_j^3+3\sum\lambda_jc_j^2)^2}\end{aligned} \tag{4.43}$$

If the slow drift approximation obtained by Naess is applied, the parameters in Eq.(4.43) should be replaced by $\tilde{\delta}_1 = 2\delta_1$, $\tilde{\nu}_1 = 2\nu_1$, and $\tilde{\theta}_1 = \theta_1$. Thus the p.d.f. of Z_1 can be approximately evaluated in the following form:

$$p_{Z_1}(x) \simeq p_\gamma(x,\tilde{\delta}_1,2\tilde{\theta}_1;\tilde{\nu}_1/2) \tag{4.44}$$

This becomes the third order approximation of p_{Z_1} because the first, second, and third order moments completely agree with the actual ones. Equation (4.44) can be exactly expanded by the generalized Laguerre polynomials as follows:

From Eq.(4.42) the c.f. of Z_1 is given by the expansion form as:

$$\begin{aligned}\phi_{Z_1} &= \phi_\gamma(u,\tilde{\delta}_1,2\tilde{\theta}_1;\tilde{\nu}_1/2)\exp[\sum_{n=4}^{\infty}A_n\xi_1^n]\\ &= \phi_\gamma\sum_{k=0}^{\infty}B_k\xi_1^k\end{aligned} \tag{4.45}$$

where $B_0 = 1$, $B_1 = B_2 = B_3 = 0$.

Using the following relation

$$\frac{\partial^k}{\partial\tilde{\theta}_1^k}\phi_\gamma = \tilde{\nu}_1(\tilde{\nu}_1+2)\cdots(\tilde{\nu}_1+2k-2)\xi_1^k\phi_\gamma \tag{4.46}$$

results in

$$\phi_{Z_1} = \sum_{k=0}^{\infty}\frac{B_k}{\tilde{\nu}_1(\tilde{\nu}_1+2)\cdots(\tilde{\nu}_1+2k-2)}\frac{\partial^k}{\partial\tilde{\theta}_1^k}\phi_\gamma \tag{4.47}$$

The partial derivative of ϕ_γ with respect to $\tilde{\theta}_1$ can also be represented in another form by:

$$\frac{\partial^k}{\partial\tilde{\theta}_1^k}\phi_\gamma = \frac{(-1)^ke^{i\tilde{\delta}_1u}}{\tilde{\theta}_1^k\Gamma(\tilde{\nu}_1/2)}\int_0^\infty e^{2iu\tilde{\theta}_1x}\frac{\partial^k}{\partial x^k}(e^{-x}\,x^{\tilde{\nu}_1/2+k-1})dx \tag{4.48}$$

Using the generalized Laguerre polynomials,

$$L_r^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{r!}\frac{d^r}{dx^r}(e^{-x}x^{r+\alpha}) \tag{4.49}$$

where $\alpha > -1$, $r = 0, 1, 2, \cdots$, Eq.(4.48) can be rewritten as follows:

$$\frac{\partial^k}{\partial\tilde{\theta}_1^k}\phi_\gamma = \frac{(-1)^k}{\tilde{\theta}_1^k\Gamma(\tilde{\nu}_1/2)}\int_{\tilde{\delta}_1}^{\infty} k!\exp(-\frac{s-\tilde{\delta}_1}{2\tilde{\theta}_1})(\frac{s-\tilde{\delta}_1}{2\tilde{\theta}_1})^{\tilde{\nu}_1/2-1} L_k^{(\tilde{\nu}_1/2-1)}(\frac{s-\tilde{\delta}_1}{2\tilde{\theta}_1})e^{ius}\frac{ds}{d\tilde{\theta}_1} \tag{4.50}$$

Finally we obtain the complete form of the p.d.f. of Z_1 in the following series form

$$\begin{aligned} p_{Z_1} &= p_\gamma(x, \tilde{\delta}_1, 2\tilde{\theta}_1; \tilde{\nu}_1/2)[1 + \sum_{k=4}^{\infty} B_k \frac{(-1)^k k!\Gamma(\tilde{\nu}_1/2)}{\tilde{\theta}_1^k\Gamma(\tilde{\nu}_1/2+k)} \\ &\times L_k^{(\tilde{\nu}_1/2-1)}(\frac{x-\tilde{\delta}_1}{2\tilde{\theta}_1})] \end{aligned} \tag{4.51}$$

where

$$B_k = \frac{\tilde{\theta}_1^k}{(-1)^k}E[L_k^{(\tilde{\nu}_1/2-1)}(\frac{x-\tilde{\delta}_1}{\tilde{\theta}_1})] \tag{4.52}$$

This final expansion form is not used except in the cases where the moments higher than third order are of importance.

The p.d.f. of $-Z_2$, as well as that of Z_1, can be also approximated by a three parameter Gamma p.d.f.($\tilde{\theta}_2$, $\tilde{\nu}_2$, and $\tilde{\delta}_2$) as follows:

$$p_{Z_2}(x) \simeq p_\gamma(x, \tilde{\delta}_2, 2\tilde{\theta}_2; \tilde{\nu}_2/2) \tag{4.53}$$

The results of Eqs.(4.44) and (4.53) indicate that the total second order response process $X(t)$ can be approximated by the difference of the two independent random variables which yield a Gamma distribution with three parameters. From the convolution integrals of the Gamma p.d.f.'s the p.d.f. of the total second order response can be obtained by:

$$p_X(x) = \begin{cases} f(\tilde{\theta}_1, \tilde{\theta}_2; \tilde{\delta}_1, \tilde{\delta}_2)\int_0^\infty (z + x - \tilde{\delta}_1 + \tilde{\delta}_2)^{\tilde{\nu}_1/2-1} z^{\tilde{\nu}_2/2-1} e^{-az}dz & \\ \times \exp(-\frac{x-\tilde{\delta}_1+\tilde{\delta}_2}{2\tilde{\theta}_1}) & x \geq \tilde{\delta}_1 - \tilde{\delta}_2 \\ f(\tilde{\theta}_1, \tilde{\theta}_2; \tilde{\delta}_1, \tilde{\delta}_2)\int_0^\infty (z - x + \tilde{\delta}_1 - \tilde{\delta}_2)^{\tilde{\nu}_2/2-1} z^{\tilde{\nu}_1/2-1} e^{-az}dz & \\ \times \exp(\frac{x-\tilde{\delta}_1+\tilde{\delta}_2}{2\tilde{\theta}_2}) & x < \tilde{\delta}_1 - \tilde{\delta}_2 \end{cases} \tag{4.54}$$

where

$$f(\tilde{\theta}_1,\tilde{\theta}_2;\tilde{\delta}_1,\tilde{\delta}_2)=\frac{1}{(2\tilde{\theta}_1)^{\tilde{\nu}_1/2}(2\tilde{\theta}_2)^{\tilde{\nu}_2/2}\Gamma(\tilde{\nu}_1/2)\Gamma(\tilde{\nu}_2/2)},$$

$$a=\frac{1}{2\tilde{\theta}_1}+\frac{1}{2\tilde{\nu}_2} \tag{4.55}$$

and the multiple-valued integrants take a principal value.

(iv) Convergence to Gaussian p.d.f

When the eigenvalues λ_j are very small compared with c_j, i.e. when λ_j are neglected, the total second order response process, $X(t)$ certainly approaches Gaussian. In this section we shall show this fact from the present theory.

From Eq.(4.43) it is found that

$$\begin{aligned}\tilde{\theta}_1 &= \frac{\sigma_X}{\sqrt{2\tilde{\nu}_1}}\\ \tilde{\delta}_1 &= \overline{X}-\sqrt{\frac{\tilde{\nu}_1}{2}}\sigma_X\end{aligned} \tag{4.56}$$

Namely the parameters of the Gamma p.d.f. are not mutually independent, two parameters in the three can be represented by the rest if σ_X and $\overline{X}$ are fixed. Taking $\tilde{\nu}_1$, which represents the degree of freedom of the Gamma p.d.f., as an independent parameter, replacing the variable x by z like

$$z=\frac{x-\overline{X}}{\sigma_X}$$

and setting $n=\tilde{\nu}_1/2$, the Gamma p.d.f. can be rewritten from Eq.(4.37) as:

$$p_\gamma(z)=\begin{cases}\frac{\sqrt{n}}{\Gamma(n)}(\sqrt{n}z+n)^{n-1}\exp(-\sqrt{n}z-n) & \text{for } z>-\sqrt{n}\\ 0 & \text{for } z\le-\sqrt{n}\end{cases} \tag{4.57}$$

When $\lambda_j \ll 1$, i.e. $n \gg 1$, we shall consider the asymptotic behavior of Eq.(4.57) as $n \to \infty$.

Noting that the first term of asymptotic expansion of Gamma function $\Gamma(z)$ is given by

$$\Gamma(z+1)\simeq\sqrt{2\pi}z^{z+1/2}e^{-z} \qquad \text{as } z\to\infty$$

and the Taylor expansion of $\log(1+u)$ for $|u|<1$ is represented as:

$$\log(1+u)=u-\frac{1}{2u^2}+o(u^2)$$

then we have:

$$\log p_\gamma \quad =-\log\sqrt{2\pi}+\{-\frac{1}{4n^2}+\frac{1}{3n^2}+\cdots\}-\frac{z^2}{2}+\frac{z^3}{3\sqrt{n}}$$

$$-(\frac{z}{\sqrt{n}}+\frac{z^2}{2n}+\cdots)+o(1) \tag{4.58}$$

If $n \to \infty$, from equation (4.58) it is found that

$$\log p_\gamma \sim -\log\sqrt{2\pi} - \frac{z^2}{2} + O(\frac{z}{\sqrt{n}})$$

that is

$$p_\gamma \sim \frac{1}{\sqrt{2\pi}}\exp(-\frac{z^2}{2}) \qquad \text{as } n \to \infty \tag{4.59}$$

This implies that p_γ can be approximated by the Gaussian p.d.f. when n is sufficiently large. But we should note that the the range which the Gamma p.d.f can be regarded as the Gaussian p.d.f. is limitted to the variable range of $z < \sqrt{n}$.

4.1.2 Maxima p.d.f.

Statistical prediction of the maxima of a random process is usually performed using the Rayleigh distribution under the condition that a random process is a stationary, narrow banded, Gaussian process with zero mean. But in the case of a second order response for a moored floating structure, this condition may no longer be satisfied. In order to exactly obtain the maxima p.d.f. of a nonlinear response, the expected number of maxima greater than a specified level is required as shown by Lin[12].

First, according to Lin, we shall show the exact theory.

Figure 4.1 is an explanatory sketch of a random process $X(t)$ for which the maxima(or minima) could be anywhere in the range of $(-\infty, \infty)$ and several maxima(or minima) could occur during one cycle as defined by mean crossings. Here, maxima are defined as peaks which satisfy the condition $\dot{X}(t) = 0$ and $\ddot{X}(t) < 0$. Whereas minima are defined as troughs satisfying the condition $\dot{X} = 0$ and $\ddot{X} > 0$. As shown in Fig.4.1 maxima and minima can take both negative and positive values. The magnitude of the maxima with positive values $\{X(t) > 0, \dot{X} = 0, \ddot{X} < 0\}$ or the minima with negative values $\{X(t) < 0, \dot{X} = 0, \ddot{X} > 0\}$ would be critical if they exceed a certain value, and hence the statistical extreme values of these maxima and the minima provide valuable information for the engineering design purpose.

For the problem of a mooring system the positive maxima are the most important, if the direction drifted by waves is positive. Since the statistical properties of negative minima can be estimated from those of positive maxima by means of the transform of random variables, the positive maxima are considered in the following analysis.

It can be assumed that $X(t)$ is stationary and zero mean without loss of generality. Then the expected number of maxima above a specified level $X(t) = \xi$, denoted as $E[M(\xi)]$, is obtained by:

$$E[M(\xi)] = \int_\xi^\infty \int_{-\infty}^0 |\ddot{x}| p_{X\dot{X}\ddot{X}}(x, 0, \ddot{x}) d\ddot{x} \tag{4.60}$$

The total expected number of maxima with positive values, denoted as $E[M(-\infty)]$, becomes

$$E[M(-\infty)] = \int_{-\infty}^{\infty} dx \int_{-\infty}^{0} |\ddot{x}| p_{X\dot{X}\ddot{X}}(x, 0, \ddot{x}) d\ddot{x} \qquad (4.61)$$

where $p_{X\dot{X}\ddot{X}}$ is the joint p.d.f.

Huston & Skopinski[13] has assumed that the ratio of their two expected numbers is approximately equivalent to the probability in which the maximum values exceed a level y, i.e. $E[M(y)/M(-\infty)] \simeq E[M(y)]/E[M(-\infty)]$. Under this assumption the probability in which the maximum positive values exceed a level y becomes

$$P_p = 1 - E[M(y)/M(-\infty)] \simeq 1 - E[M(y)]/E[M(-\infty)] \qquad (4.62)$$

Then maxima p.d.f. is given by:

$$p_p(y) = -\frac{1}{E[M(-\infty)]} \int_{-\infty}^{0} \ddot{x} p_{X\dot{X}\ddot{X}}(y, 0, \ddot{x}) d\ddot{x} \qquad (4.63)$$

In the case that $X(t)$ is the Gaussian process, p_p has already been obtained by Cartwright & Longuet-Higgins[14]. It can be prescribed by two parameters, i.e. spectrum band width parameter and variance. As well known, when the band width parameter is close to 1, i.e. wide banded process, p_p approaches the Gaussian p.d.f., and when the parameter close to 0, p_p approaches the Rayleigh p.d.f.

But statistical characteristics and maxima p.d.f. of nonlinear responses has not been found out yet. So we must introduce some approximations to obtain p_p for the nonlinear response.

For this purpose the following assumptions are introduced.

(1) The response is narrow banded, i.e. the negative maxima and positive minima are negligible.

(2) The response is stationary.

(3) The expected number of crossings at a specified level with a positive gradient is equal to that of maxima over it, i.e. one-to-one correspondence between zero-upcrossings and maxima.

Assumption (1) imposes considerable limitations to our objective. However the condition is usually satisfied, except for fatigue analysis, because if the specified level is sufficiently high, the negative maxima or positive minima that exist over this level are infrequent. In general, using these assumptions, the maxima (or minima) probability is overestimated as compared with exact one because the expected number of maxima over a specified level is always greater than those crossing that level. Since statistical properties of the minima can be obtained from those of the maxima, by means of a variable transform, only the maxima will be considered in the following analysis.

Using the above assumptions, then

$$M(y) \simeq N^{+}(y) \tag{4.64}$$

where N^{+} is a random number crossing a specified level y at positive gradient and its expectation per unit time is given by:

$$E[N^{+}(y)] = \frac{1}{2}\int_{-\infty}^{\infty} |\dot{x}| p_{X\dot{X}}(y, \dot{x}) d\dot{x} \tag{4.65}$$

Thus a p.d.f. for an event where the maxima are greater than a level $y + \overline{X}$ is given by:

$$p_p(y) = -\frac{d}{dy}\left\{\frac{\int_0^{\infty} p_{X\dot{X}}(y + \overline{X}, \dot{x})\dot{x}d\dot{x}}{\int_0^{\infty} p_{X\dot{X}}(\overline{X}, \dot{x})\dot{x}d\dot{x}}\right\} \tag{4.66}$$

where $p_{X\dot{X}}$ is a joint probability density function of the response X and its time derivative $\dot{X}$.

In this way, under narrow band assumption the problem obtaining the maxima p.d.f. of nonlinear response can be transformed to the problem obtaining the joint p.d.f. $p_{X\dot{X}}$.

(i) Series approximate solution

Obviously if instantaneous p.d.f.'s can be expanded into useful series representations, one would expect that similar useful generalized expansions would also exist for higher dimension p.d.f.'s.

A particularly useful expansion for our purpose was introduced by the autors[6]. The following development closely follows their original works.

Let $p(x_1, x_2)$ be a joint p.d.f. for the variables x_1 and x_2. The corresponding instantaneous p.d.f.'s are then

$$\begin{aligned} p_1(x_1) &= \int p(x_1, x_2) dx_2 \\ p_2(x_2) &= \int p(x_1, x_2) dx_1 \end{aligned} \tag{4.67}$$

Using the instantaneous p.d.f.'s as weighting functions, we can construct two sets of orthonormal polynomials $\{\Lambda_{1n}(x_1)\}$ and $\{\Lambda_{2n}(x_2)\}$ from the integral relation

$$\begin{aligned} &\int p_1(x_1)\Lambda_{1m}(x_1)\Lambda_{1n}(x_1)dx_1 = \delta_{mn} \\ &\int p_2(x_2)\Lambda_{2m}(x_2)\lambda_{2n}(x_2)dx_2 = \delta_{mn} \end{aligned} \tag{4.68}$$

If we assume that it is permissible to expand $p(x_1, x_2)$ in terms of those two sets of orthonormal functions, then

$$p(x_1, x_2) = p_1(x_1)p_2(x_2)\sum_{m,n} a_{mn}\Lambda_{1m}(x_1)\Lambda_{2n}(x_2) \tag{4.69}$$

By employing Eq.(4.68) in Eq.(4.69), we can evaluate the expansion coefficients,

$$a_{mn} = \int\int p(x_1, x_2)\Lambda_{1m}(x_1)\Lambda_{2n}(x_2)dx_1dx_2 \tag{4.70}$$

If the matrix (a_{mn}) is diagonal, i.e. $a_{mn} = a_n\delta_{mn}$,

$$p(x_1, x_2) = p_1(x_1)p_2(x_2)\sum_{n} a_n\Lambda_{1n}(x_1)\Lambda_{2n}(x_2) \tag{4.71}$$

This is equivalent to the Mercer expansion[15) of the kernel function in the integral equation (see Appendix E).

The validity of Eq.(4.71) can be illustrated as follows:

Let $p(x_1, x_2)$ be the joint Gaussian p.d.f. as

$$p(x_1, x_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\varrho^2}}\exp\{-\frac{(x_1^2 + x_2^2 - 2\varrho x_1x_2)}{2\sigma^2(1-\varrho^2)}\} \tag{4.72}$$

with corresponding p.d.f.

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{x^2}{2\sigma^2}) \tag{4.73}$$

Using the Mehler's expansion[16) given by

$$\frac{1}{\sqrt{1-u^2}}\exp[-\frac{\{u^2(x_1^2 + x_2^2) - 2ux_1x_2\}}{2(1-u^2)}]$$

$$= \sum_{n=0}^{\infty}\frac{u^n}{n!}H_n(x_1)H_n(x_2) \tag{4.74}$$

where $H_n(x)$ is the Hermite polynomials of order n, and inserting Eq.(4.74) into (4.72),we get

$$p(x_1, x_2) = \frac{1}{2\pi\sigma^2}\exp\{-\frac{x_1^2 + x_2^2}{2\sigma^2}\}\sum_{n=0}^{\infty}\frac{\varrho^n}{n!}H_n(\frac{x_1}{\sigma})H_n(\frac{x_2}{\sigma}) \tag{4.75}$$

Since the matrix a_{mn} in Eq.(4.70) is not always diagonal in general cases, the joint p.d.f. $p_{X\dot{X}}$, the first approximation of which is the joint Gaussian p.d.f., may be expressed as:

$$p_{X\dot{X}}(x,\dot{x}) = \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}}\exp\{-\frac{(x-\overline{X})^2}{2\sigma_X^2} - \frac{\dot{X}^2}{2\sigma_{\dot{X}}^2}\} \times \sum_{m,n} b_{mn} H_m(\frac{x-\overline{X}}{\sigma_X}) H_n(\frac{\dot{x}}{\sigma_{\dot{X}}}) \tag{4.76}$$

where $\overline{X}$ is the mean of X and b_{mn} is a function of the higher moments of X and $\dot{X}$.

Vinje[17] has found the same equation as (4.76) by using the Taylor expansion of cumulants. Hineno[18] and Dalzell[19] extended the above method to the method obtaining three dimensional joint p.d.f.'s, i.e. $p_{X\dot{X}\ddot{X}}$.

(ii) Independence approximation

Although X and $\dot{X}$ are not generally mutually independent, let their independence be assumed. Then a p.d.f. of maxima that are greater than $y+\overline{X}$ is given by:

$$p_p(y) = -\frac{d}{dy}\{\frac{p_X(y+\overline{X})}{p_X(\overline{X})}\} \quad , y \geq 0 \tag{4.77}$$

This means that the maxima p.d.f. can be represented in terms of the derivative of the p.d.f. of the instantaneous response.

4.1.3 1/n th highest mean amplitude and extreme value

From the maxima p.d.f., 1/n th highest mean value can be represented as:

$$\overline{X}_{\frac{1}{n}} = \int_{\overline{X}_{\frac{1}{n}}}^{\infty} x p_p(x) dx \tag{4.78}$$

$$1/n = 1 - P_p(\overline{X}_{\frac{1}{n}}) \tag{4.79}$$

where P_p is the peak probability distribution function.

An extreme value will be derived by applying the order statistics. The extreme value is defined here as the largest maxima that occur in N observations.

Let $(\eta_1, \eta_2, \cdots, \eta_N)$ be an ordered sample of size N, where η_i have the same p.d.f. given by Eq.(4.66). If η_i is recorded as $\eta_1, \eta_2, \cdots, \eta_N$, η_i can be regarded as the output of an independent random variable z_i. Thus the random variable z_N, which is the largest η_N in the ordered sample, has the following p.d.f.:

$$f(z_N; N) = N p_p(z_N)[1 - P_p(z_N)]^{N-1} \tag{4.80}$$

Then the estimation of an extreme response is obtained as:

$$E[z_N] = \int_0^{\infty} Z \cdot f(Z; N) dZ \tag{4.81}$$

Approximation based on Poisson distribution law

Naess[5)] has introduced an alternative approximation based on Poisson distribution law to obtain the extreme statistics. His approximation is as follows:

The statistics of high level excursions and extreme values of the total second order response are largely determined by the mean upcrossing frequency $\nu_Z^+ = E[N^+(z)]$ for large z. If extreme values are associated with very high levels and upcrossings of such levels are rare events, then the probability such that the extreme values, i.e. $\hat{Z}(T) = max\{X(t) : T \geq t \geq 0\}$, is less than any level z is given by:

$$P_{rob}\{\hat{Z}(T) \leq z\} = \exp(-\nu_Z^+ T) \quad \text{as } z \to \infty \qquad (4.82)$$

where T is an observation time. This leads to the assumption that these upcrossings are statistically independent, which in term implies the Poisson probability law. Except in the case of narrow banded process, this would be a reasonable approximation. Now considering the expected value as a statistical measure of the extreme value, its expectation is given as:

$$E[\hat{Z}(T)] = \int_0^\infty z dP_{\hat{Z}}(z) \qquad (4.83)$$

where $P_{\hat{Z}}(z) = P_{rob}\{\hat{Z}(T) \leq z\}$.

Since the number of observations N can be replaced by $N = \nu_0^+ T$, we get:

$$\begin{aligned} \log[(1 - P_p(z))^N] &= N \log(1 - \frac{\nu_Z^+ T}{N}) \\ &= -\nu_Z^+ T + O(\frac{\nu_Z^+ T}{N}) \end{aligned} \qquad (4.84)$$

This implies that $(1 - P_p)^N$ approaches $\exp(-\nu_Z^+ T)$ as $N \to \infty$. That is, Eq.(4.81) tends to Eq.(4.83) when $N \to \infty$. Thus it is expected that both Eq.(4.81) and Eq.(4.83) lead to a same extreme value estimate for a large N.

4.2 Numerical Examples

From this point forward it will be assumed that the rapidly varying part of the pure second order response is negligible. In this case, the Naess' method does then yield a complete analytical solution for the pure second order response, but it can not be applied to the problem of obtaining the total second order response p.d.f. unless the linear response is negligibly small. Unfortunately an exact closed form or numerical solution for this case has not yet been found. The direct approach to the problem by approximating the p.d.f. using a power series would probably theoretically work, but the effort involved is considered too great. The logical and most conservative approach is to attempt to utilize only a few terms of series expansion. Experience dictates that an important step in series solution techniques lies in the choice of an expansion function which closely represents the desired nonlinearity characteristics with a minimum number of terms. The present method, "*new approximate theory* " is a series expansion approach that approximates the total second order response p.d.f. by three terms of the generalized Laguerre expansion. This method also gives an approximate solution for the pure second order response. So exactly speaking, the method is a third order approximation because the first, second, and third order statistical moments completely agree with the exact ones. Additionally a convolution integral has to be conducted in the present method which is not the case for the exact Naess' solution. Thus it should not be inferred that the present method is more efficient than the exact Naess' solution. The present method will however be effective in evaluating the effect of the statistical interference between first and second order responses for the extreme response.

(1) Investigation to the pure second order forces and responses

In this section, the present method will be compared to the exact Naess' solution for pure second order forces and sway motion responses in order to show that the present method is an accurate enough approximation. The moored floating structures that will be used for comparisons are two dimensional, lie in the horizontal plane, and have linear restoring forces. The half submerged circular structure has a diameter of 20m, and the half submerged rectangular structure has beam to draft ratio of 2. The principal dimensions are given in Table 4.1. In order to compare the present method for pure second order forces with the Naess' method, the quadratic transfer function $G_2^f(\omega_1, -\omega_2)$ of slow drift forces is required. Thus the same numerical estimates used by Naess were utilized (Faltinsen and Løken[20]).

Tables 4.2 and 4.3 indicate the numerical estimates of the quadratic transfer functions that were obtained by Faltinsen and Løken. To specify the sea state an International Ship Structure Congress (ISSC) spectrum with a significant wave height $H_s = 2m$ and an average period $T_1 = 5.5sec$ is used and is given by:

$$\hat{S}_\zeta(\omega) = \frac{173H_s^2}{T_1^4\omega^5}\exp(-\frac{691}{T_1^4\omega^4}) \tag{4.85}$$

Using this data as a basis, the eigenvalue problem was numerically solved by Naess[5].

Figure 4.2 (a) compares the p.d.f. obtained from the present method and the exact one for the half circular structure, and Fig.4.2 (b) indicates the same comparison for the rectangular cylinder. The results of the present method closely agree with the exact ones except in the peaked area. The difference in the vicinity of the peak may be attributed to the difference between the exact higher order moments greater than the third order, and the ones obtained from the present method.

A comparison of both methods for the pure second order motions will be presented next.

First consider the linear dynamic system as:

$$\ddot{X}_2 + 2\kappa\omega_0\dot{X}_2 + \omega_0^2 X_2 = \frac{F^{(2)}(t)}{M} \tag{4.86}$$

where $F^{(2)}(t)$ is the slowly varying drifting force, $X_2(t)$ the corresponding slow drift sway response, κ a relative damping coefficient , ω_0 the undamped natural frequency, and M the total mass including an added mass per unit length of the cylinders. Parameter values for κ, ω_0, and M are given in Table 4.1. The linear transfer function $H_L(\omega)$, which corresponds to equation (4.68), is given by:

$$H_L(\omega) = \frac{1}{(\omega_0^2 - \omega^2) + 2i\kappa\omega_0\omega} \tag{4.87}$$

Thus, the quadratic transfer function of the slow drift sway response can be represented by:

$$G_2(\omega_1, -\omega_2) = \frac{H_L(\omega_1 - \omega_2)G_2^f(\omega_1, -\omega_2)}{M} \tag{4.88}$$

The same input wave spectrum given in Eq.(4.85) was used for calculating the eigenvalues for the sway response. Naess calculated only eight eigenvalues. This is equivalent to assuming that a random seastate has only eight frequency components. This number is insufficient if a practical seastate situation is considered. Furthermore Naess' results appear to be too inaccurate to estimate eigenvalues for a lightly damped oscillator since the amplitude of H_L changes suddenly at $|\omega_1 - \omega_2| \simeq \omega_0$. As a result, the authors[21] extended the quadratic transfer functions given in Tables 4.2 and 4.3 to higher dimensional matrices by interpolation, then solved the eigenvalue problems, and investigated the relationship between the variances between the pure second order responses and the dimension of the quadratic transfer matrices. From this it was determined that the variances of pure second order response change largely with a decrease of the dimension, and that at least a dimension greater than 200 is required for getting stable variances. Thus based on the above determination 200 dimensions of the quadratic transfer matrices were used.

Figures 4.3 (a), (b), (c) show respectively the p.d.f.'s of pure second order sway motion responses, their tail behavior, and their 1/n th highest mean values for case 1 of Table 4.1. Figures 4.4 (a), (b), (c) show the same parameters for case 2, and Figs. 4.5 (a), (b), (c) case 3. These figures indicate that the p.d.f. calculated by the present method is in good agreement with Naess' exact p.d.f. in contrast to the differences in the pure second order force responses of Fig.4.2 which were discussed previously. There is however a noticeable difference in the tail of the p.d.f. shown in Figs. 4.3 (b), 4.4 (b), and 4.5 (b). The effect of this difference is small because the difference in the 1/n th highest mean amplitude by the present method shown in Figs. 4.3 (c), 4.4 (c), and 4.5 (c) is only slightly lower than the exact one, i.e. a difference of less than 3%. This difference becomes small as the damping coefficient decreases, i.e. the difference between the Naess' and the present methods in Fig.4.3 (c) is smaller than 1%. Therefore it is considered from practical point of view that the present method is a good approximation with a high degree of accuracy.

The difference between the p.d.f. of pure second order responses and the Gaussian p.d.f. with equal mean and variance will briefly be discussed next. Before doing this it should be noted that the Rayleigh method in figures 4.3 (c), 4.4 (c), and 4.5 (c) is an approximation to predict the 1/n th highest mean amplitude under the assumptions that responses are Gaussian and narrow banded, i.e. the maxima p.d.f. is a Rayleigh p.d.f. When the damping coefficient κ is significantly reduced to a value of 3×10^{-5}, it can be seen by comparing Fig.4.3 (a) to Figs.4.4 (a) and 4.5 (a) that the mean value, which is the mean drift displacement, is small. Similarly the asymmetry of the p.d.f. about the mean value is small, indicating that the pure second order response p.d.f. approaches the Gaussian p.d.f. However the difference becomes much more significant in the tail response as well as the 1/n th highest mean amplitude. This is as expected because the tail of the pure second order response p.d.f. behaves like $O(\exp(-x))$(e.g. Eq.(4.14)), while that of a Gaussian p.d.f. behaves like $O(\exp(-x^2))$. When κ is increased there is an increase in the mean value and the asymmetry of the p.d.f. around the mean value. Thus as the damping coefficient is increased there is a greater deviation between the p.d.f. of the pure second order response and the Gaussian p.d.f. This results in Gaussian approximation that will significantly underestimate high level excursions and extreme responses. The use of moored circular or rectangular structures shows no differences and thus do not influence this conclusion.

(2) Statistical interference between first and second order responses

In general the first and second order responses are not mutually independent so it is important to study the statistical interference of both responses. Thus we shall consider the following system:

$$\ddot{X} + 2\kappa\omega_0\dot{X} + \omega_0^2 X = \frac{(F^{(1)}(t) + F^{(2)}(t))}{M} \tag{4.89}$$

where $F^{(1)}$ is a linear wave exciting force, $M = 3.21 \times 10^5$ kg/m, ω_0 =0.1 rad/sec, and the damping coefficient κ being equal to 0.1,0.006, and 0.0001.

Calculations were conducted only for the half circular cylinder. The wave exciting forces were calculated based on two dimensional potential theory(see Table 4.4). The ratio of the standard deviation of the second order exciting force response to the first order response (σ_2/σ_1) is 3.31×10^{-4}, and the ratios for the sway motion response are 1.36, 2.9, and 4.96 for κ=0.1, κ=0.006 and κ=0.0001, respectively. The numerical results are shown in Figs.4.6 through

4.8, and are compared to the Gaussian p.d.f. and the p.d.f. for pure second order responses. Based on these figures, it was determined that the p.d.f. of the total second order response was widely distributed, while that of the pure second order response was narrowly distributed, with the Gaussian p.d.f. being located between these two distributions. The width of the p.d.f. of the total second order response is strongly dependent on the damping coefficient. When the damping coefficient is decreased, the width of the p.d.f. of the total second order response becomes narrow and approaches that of the pure second order response. The difference between the p.d.f.'s. of the pure and total second order responses in the tail region may be caused by the following reasons:

> Since maximum double amplitudes of a pure first order response can possibly occur at the pure second order response peaks, the probability density of the total second order response increases as compared to the pure second order tail response.

Furthermore it should be noted that the p.d.f. of total second order response differs from the Gaussian p.d.f. in the tail region even though both p.d.f.'s are, on the whole, in good agreement as the damping force decreases to zero. With respect to the $1/n$ th highest mean amplitude, the results shown in the total second order response are the largest of the three responses and significantly deviate from the well-known expected value that is estimated using the assumption that the peak p.d.f. is a Rayleigh p.d.f. when the damping coefficient is increased . Thus, if the pure second order approximation is used to predict the highest mean values of the total second order responses or if the assumption that the peak p.d.f. is a Rayleigh p.d.f. are applied, this will cause a large underestimation of high level excursions and extreme values. This fact was experimentally confirmed by the authors[9].

The statistical interference between the first and second order responses can be significantly large as shown by the use of these examples, and so it must be taken into account for the motion prediction of moored vessels in random seas.

(3) Relationship between the damping and restoring force coefficients and 1/10 th highest mean amplitude

In this section the variation of the 1/10 th highest mean amplitudes is investigated following changes in damping and restoring force coefficients. Fig.4.9 (a) shows the relationship between the damping coefficient and the 1/10 th highest mean amplitude. In this figure all the lines approach the well -known expected value for the Rayleigh p.d.f. as the damping coefficient is decreased, but the results of the total second order response deviate considerably from the estimated one with an increase in the damping coefficient.

The relation between the restoring force coefficient and the 1/10 th highest mean amplitude is shown in Fig.4.9 (b). In this figure the X axis indicates the undamped natural frequency because the restoring force coefficient is proportional to the square of the natural frequency if the total mass is held constant. When the restoring force is increased the 1/10 th highest mean amplitude of the pure second order response approaches the well-known expected value for the Rayleigh p.d.f., while the 1/10 th highest mean amplitude of the total second order response deviates from its expected value for Rayleigh p.d.f. by becoming larger.

4.3 Comparisons between estimates and experimental results

In order to investigate the applicability of the present method to the measured slow drift motion, we shall compare the results estimated by the present method with the statistics obtained from the model test(see 3.4.1).

(1) Instantaneous p.d.f.

First of all, we must solve the eigenvalue problem (4.5) for obtaining the instantaneous p.d.f. Utilizing the quadratic transfer function with viscous effect shown in 3.4.5, the integral equation leads to the linear algebraic equations with 512 dimensions since the lag number of the wave spectrum was 256. However if we adopt the slow drift approximation indicated by Naess, the integral equation generates a set of double eigenvalues. Thus the algebraic equations can be reduced to a set of 256 frequencies in the positive frequency range. In the 256 frequencies we use only the 32 frequency components which are within a frequency range where the wave spectral densities are more than 10 % to the peak.

Table 4.5 shows the examples of eigenvalues obtained by solving the 32 dimensional algebraic equations.

Comparisons between the statistical values estimated from the relation (4.11) and the sample ones obtained from the time average of the measured data are shown in Table 4.6, where $\tilde{\theta}_i$ and $\tilde{\nu}_i$ are parameters of Gamma p.d.f. and "*wave condition No.*" indicated in the tables corresponds to the number shown in Table 3.3. From both tables it is seen that the estimated statistical values agree with the sample ones even though the number of eigenvalues used for calculation is a few.

The instantaneous p.d.f.'s of slowly varying second order surge response are indicated in Figs.4.10 and 4.11. In these figures the solid line shows the line due to the present method, the dash-dotted line expresses the Gaussian distribution function and the broken line the result of the third order Gram-Charlier expansion. The probability distribution is asymmetry with respect to the mean value even if the restoring force is linear, and it has the tendency that the tail spreads towards the direction drifted by waves. And the difference between the probability distribution due to the present method and the Gaussian distribution is certainly significant at the tail and the agreement of the present method and the third order Gram-Charlier series method with the observed histograms is still good.

(2) Maxima p.d.f.

For mooring design purpose, positive maxima is the most important of all maxima. Figure 4.12 compares the observed positive maxima and the estimated maxima p.d.f.'s. The dash-dotted line is the Rayleigh p.d.f., the solid line is the curve due to the present method, and the broken line is the result due to the third order Gram-Charlier series method, where an assumption of the independence between the response process and its time derivative process was used for comparison. From this figure, it is found that the observed positive maxima histograms exponentially spread towards the tail and that the estimated p.d.f.'s due to the present method are in rough agreement with the observed ones.

(3) Extreme response

Comparisons between the extreme responses due to the present method and the maximum excursions in N_p observations in the total measured data are shown in Figs.4.13 and 4.14. In these figures the dash-dotted line indicates the estimation results by Longuet-Higgins' method[14], which uses the assumption that the maxima p.d.f. yields the Rayleigh p.d.f., and the black circles represent the largest values in each observations of maxima in the long measured data, the broken line is the result due to the third order Gram-Charlier series method, and the solid line is the estimate due to the present method. The extreme values are normalized by the standard deviation of the response. From these figures it is found that the results from the Longuet-Higgins' method significantly underestimate the extreme values whereas those from the present method show fairly good agreement with the largest excursions in the measured data, which are samples of the extreme values.

REFFERENCES IN CAPTER 4

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Chapter 5

Conclusions

This paper describes the researches about the slowly varying second order response simulations of moored floating structures in random seas and its stochastic analysis.

First, we reviewed the study on the slowly varying drift forces causing the slowly varying response and discussed four problems excluded in the investigations obtained up to now. As the most important problems in them, the following problems are treated in this paper.

a) Hydrodynamic forces of slow drift motion in still water are modified in waves.

b) The Newman-Pinkster's approximation for the slowly varying drift force does not satisfy the condition of physical causality.

Second, it is shown that the total second order force including slow drift forces can be represented by a two term Volterra functional series. Physical meanings of the kernel functions in the functional series are investigated from a viewpoint of frequency response functions (or transfer functions) and a method estimating the kernel ones from experimental data is also studied, which is the method using the bispectrum (a kind of higher order spectra). Furthermore a new functional model such that the second term of the Volterra functional series can be represented by the equivalent linear process of instantaneous wave power is developed. The new function model is based on the Wiener filter theory.

Several kinds of experiments have been carried out. Relation between the kernel function and the frequency response function of the slow drift force is investigated through comparisons between the experimetal results and numerical calculations. And the applicability of the newly developed functional model is studied by comparing between the experimental data and numerical simulations. And the unsolved problems a)(i.e. how much the hydrodynamic forces in still water are modified in waves) and b) are investigated by using the new functional model.

Finally, on the basis of the obtained results a theory of probability density functions(p.d.f.'s) is developed for an instantaneous total second order response and its maxima, in order to predict 1/n th highest mean amplitudes and extreme responses. New formulas for the total second order p.d.f.'s which include not only quadratic but also linear responses are derived. These new p.d.f.'s can be represented by the generalized Laguerre polynomials of which the first term is a Gamma p.d.f. consisting of three parameters. Assuming that the response and its time derivative processes are mutually independent, the 1/n th highest mean amplitude can be evaluated numerically from the derivative of the instantaneous response p.d.f.. This method is first applied to the sway motion of moored floating semi-circular and rectangular two dimensional cylinders, and the applicability of the method is studied by comparisons with Naess' exact solution. The variation of the 1/n th highest mean amplitude of the total second order response is then investigated following increases in damping and restoring forces. And comparisons between the experimental results and the calculated ones obtained from the present theory are carried out. The applicability of the present theory is confirmed.

The summary of the results obtained in this paper are as follows:

(1) The total second order responses(forces and motions) can be represented by a two term Volterra functional series and the quadratic transfer function in the second term of the functional series physically correspond to a frequency characteristic of the mean and slowly varying drift responses. On the basis of the mathematical fact that by using the Wiener filter theory, the second term of the Volterra functional series can be expressed by an equivalent linear process of instantaneous wave power in stochastic sense, a new functional model is developed. This model can be used not only to simulate mean and slowly varying drift responses of moored floating structures but also to solve the problems a) and b) mentioned previously.

(2) The quadratic transfer function in the Volterra functional series (or present functional model) can not only be estimated from the bispectral analysis of experimental data, but also be calculated from pressure integrals over the instantaneous wetted surface of a floating body within the potential theory. As to the quadratic transfer function, comparison between the result obtained through the cross bispectral analysis of experimental data and the numerical ones is conducted. As the result, it is found that the numerical result based on the potential theory is remarkably lower than the experimental ones and the difference of both results can be accounted for by viscous drift force, which occurs by the finiteness of incident wave amplitude and is proportional to the third power of wave amplitude. If the viscous drift force is taken into account to the quadratic transfer function obtained from numerical calculations even though it is approximately

evaluated, the corrected numerical result is in good agreement with the experimental one. And the linear frequency response function can roughly be estimated from the ususal linear motion prediction method considering the viscous damping force. But when the slow drift motion response is dominant compared with the linear motion response, the damping force at the slow drift motion increases by 1.6 times as large as one in still water whereas the added mass force at the slow drift motion becomes smaller than that in still water. It may be considered that for semi-submersibles this phenomenon is attributed to not only the nonlinear coupled viscous damping but also the wave drift damping and others.

(3) Comparisons between the simulated results due to the present functional model and the experimental ones have been conducted in time domain, and it has confirmed that both results are in good agreement, however it remains unsolved how much and why the added mass and the damping forces in still water are modified in waves.

(4) An approximate solution is presented for calculating the p.d.f.'s (instantaneous p.d.f. and maxima p.d.f.) of total second order responses including first order as well as second order motions. It is confirmed through comparisons with Naess' exact solution that the present method is an accurate approximation for pure second order forces and responses.

(5) Using the present method, an investigation to determine the statistical interference between the first and second order responses was conducted for a system with a linear damping and a linear restoring forces. The p.d.f. of the total second order response differs from that of the pure second order response. In fact it becomes a widely-banded distribution with an increase in the damping coefficient. Additionally it significantly deviates from the Gaussian p.d.f..

(6) The 1/10 th highest mean amplitude of the total second order response is greater than that obtained using the pure second order approximation or by using the conventional method which is estimated under the assumption that the peak p.d.f. is a Rayleigh p.d.f.. Thus the statistical interference between the first order and second order responses must be taken into account for prediction of extreme responses and high level excursions. The statistical interference changes with variations in the damping and restoring forces.

(7) As to the extreme response, comparison between the result obtained from the present method and one from the model test during long duration has been carried out. It is confirmed that the usual prediction method based on the Longuet-Higgins' method significantly underestimates the measured results while the present method estimates them very well. And

it is shown that the extreme response of the total second order response is greater than that based on the assumption of the pure second order response.

Moreover, some subjects excluded in this paper, for example, mooring forces, comparisons between estimated results and at-sea experimental results etc., are going to be completed in future.

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