APPENDICES

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Appendix A

Theory of wave drift forces based on the potential theory

The wave drift force based on the potential theory is the low frequency component of the second order force caused by nonlinear interaction between first order phenomena at multiple frequency. In order to exactly evaluate the force it is necessary to formulae the second order problem in a sophisticated manner.

In this section we introduce the regular perturbation technique formulated by Ogilvie[1].

A.1 Coordinate system

we define two sets of axes:

$Oxyz = Ox_1x_2x_3$: inertial(space fixed) axes;
$O'x'y'z' = O'x'_1x'_2x'_3$: body fixed axes.

The $Oxyz$ axes have their origin in the plane of the undisturbed free surface with z axis pointing upwards. The two sets of axes coincide when the body is at rest.(see Fig.A.1)

A.2 Boundary value problem

We consider hydrodynamic forces acting on the floating body oscillating in waves under the coordinate system shown in Fig.A.1. The theory is based on the assumption that the fluid surrounding the body is inviscid, irrotational, homogeneous, and incompressible. The fluid motions may then be described by a velocity potential from which the velocity field can be derived by taking the gradient:

$$\vec{V} = \nabla\Phi \tag{A.1}$$

where $\Phi(\vec{x}, t)$ is a velocity potential and it satisfies the Laplace equation:

$$\text{[L]: } \Delta\Phi = 0 \tag{A.2}$$

If the potential Φ is known, the pressure in a point in the fluid may be determined using the Bernoulli equation:

$$\frac{p(x, y, z, t)}{\rho} = -\Phi_t - \frac{(\nabla\Phi)^2}{2} - gz \tag{A.3}$$

where ρ is the fluid density, g is the gravitational acceleration. If the elevation of the free surface is given by $\zeta(x, y, t)$, the following two conditions must be satisfied:

$$\frac{D(z-\zeta)}{Dt} = \Phi_z - \zeta_t - \Phi_x\zeta_x - \Phi_y\zeta_y = 0 \quad \text{on } z = \zeta \tag{A.4}$$

$$\frac{p_0}{\rho} = -\Phi_t - (\nabla\Phi)^2 - gz \quad \text{on } z = \zeta \tag{A.5}$$

The first shows the kinematic condition of free surface and the second does that the pressure is constant on the free surface. These free surface conditions are exact under the assumption that viscosity and surface tension are negligible.

Eliminating ζ from the conditions, the free surface conditions can be rewritten as:

$$\begin{aligned} \text{[F]:} \quad & \Phi_{tt} + g\Phi_z + 2(\Phi_x\Phi_{xt} + \Phi_y\Phi_{yt} + \Phi_z\Phi_{zt}) + \Phi_x^2\Phi_{xx} + \Phi_y^2\Phi_{yy} + \Phi_z^2\Phi_{zz} \\ & +2(\Phi_x\Phi_y\Phi_{xy} + \Phi_y\Phi_z\Phi_{yz} + \Phi_z\Phi_x\Phi_{zx}) = 0 \qquad \text{on } z = \zeta \end{aligned} \tag{A.6}$$

Let the body surface, S, be given by an equation of the form:

$$S(x, y, z, t) = 0$$

and let $\vec{n}$ be a unit vector normal to the body surface, pointing outward from the fluid, thus into the body. A body condition given by:

$$\text{[H]: } \frac{\partial\Phi}{\partial n} = \vec{n} \bullet \nabla\Phi = v_n \tag{A.7}$$

where v_n is the normal component of velocity of the body itself.

If a bottom surface is given by $z = h(x, y)$, the bottom condition becomes:

$$\text{[B]: } \frac{\partial\Phi}{\partial n} = 0 \quad \text{on } z = h(x, y) \tag{A.8}$$

In addition, an outgoing wave radiation condition must be satisfied.

In general, it is difficult to directly solve the above boundary value problem in time domain because the free and body surfaces moves with time, and the boundary value problem must have already been solved to determine the movements of free and body surfaces. These problems can usually be solved by making a linearization by means of a perturbation technique. In order to carry out a perturbation analysis, we assume that there exists a small parameter that provides a basis for ordering all quantities that arise. We can think of this parameter as the maximum wave slope, for example, although its precise definition does not really matter. We assume further that quantities such as and can be expressed as power series in:

$$\Phi(x,y,z,t) \sim \sum \epsilon^j \varphi_j(x,y,z,t) + O(\epsilon^{N+1}) \tag{A.9}$$

$$\zeta(x,y,t) \sim \sum \epsilon^j \zeta_j(x,y,t) + O(\epsilon^{N+1}) \tag{A.10}$$

Substituting these expansions into the free surface conditions, and assuming that all quantities that are supposed to be evaluated on $z = \zeta$ can be evaluated alternatively by expansions with respect to $z = 0$, then we get the following pairs of free surface boundary conditions for the first and second order problems:

$$O(\epsilon): \qquad \varphi_{1tt} + g\varphi_{1z} = 0 \quad \text{on } z = 0 \tag{A.11}$$

$$\zeta_1 = -\frac{\varphi_{1t}}{g}\,|_{z=0} \tag{A.12}$$

$$O(\epsilon^2): \qquad \varphi_{2tt} + g\varphi_{2z} = -\frac{\partial}{\partial t}(\varphi_{1x}^2 + \varphi_{1y}^2 + \varphi_{1z}^2) + \frac{\varphi_{1t}}{g}\frac{\partial}{\partial z}(\varphi_{1tt} + g\varphi_{1x}) \text{on } z = 0 \tag{A.13}$$

$$\zeta_2 = [-\frac{\varphi_{2t}}{g} - \frac{1}{2}g(\varphi_{1x}^2 + \varphi_{1y}^2 + \varphi_{1z}^2) + \frac{\varphi_{1t}\varphi_{1tz}}{g^2}]\,|_{z=0} \tag{A.14}$$

In addition, we need a bottom condition and a radiation condition for each problem.

A.3 Body surface condition

Before considering the body surface condition, we shall define the transformation of coordinates.

Let the position of O′ with respect to O be denoted by the vector $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ and let the position vector to a point in space be denoted by

$$\vec{X} = (x, y, z) = (x_1, x_2, x_3) \tag{A.15}$$

$$\vec{X}' = (x', y', z') = (x_1', x_2', x_3') \tag{A.16}$$

respectively, in the two coordinate systems. The position vectors are related by a linear transformation

$$\vec{X}' = \mathbf{D}(\vec{X} - \vec{\xi}) \tag{A.17}$$

$$\vec{X} = \mathbf{D}^{-1}\vec{X}' + \vec{\xi} \tag{A.18}$$

where $\mathbf{D}$ is a matrix presenting the rotation of body and $\mathbf{D}^{-1}$ is its transpose. For such matrices, we note that the product $\mathbf{D} \bullet \mathbf{D}^{-1}$ is the unit matrix.

In order to consider the rotations, we may use the concept of Euler angles (or an equivalent) to specify the instantaneous orientation of the body, and the resulting expression for $\mathbf{D}$ depends on the order in which the three rotations are taken. The order is, of course, arbitrary, since any displacement of a rigid body can be described as the sum of a translation and a single rotation about some axis. But that axis constantly changing in time, and so we must use a systematic method of describing the kinematics of the body. Our choice is to take roll, pitch, and yaw, in that order. These are not the Euler angles described in a textbook, but they are more useful for our present problem.

First, neglect the translations and consider only rotations (thus O and O' coincide). Define a new coordinate system $O\tilde{x}\tilde{y}\tilde{z}$ that is identical to the $Oxyz$ system except for a positive rotation ξ_4 about the x axis. Thus $\tilde{x} = x$. The transformation from $\vec{X}$ to $\vec{\tilde{X}}$ is simple:

$$\begin{aligned} \vec{\tilde{X}} &= \mathbf{A}\vec{X} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\xi_4 & \sin\xi_4 \\ 0 & -\sin\xi_4 & \cos\xi_4 \end{pmatrix} \vec{X} \end{aligned} \tag{A.19}$$

Then we make a second rotation, this time through an angle ξ_5 about the $\tilde{y}$ axis. Let the new axes be denoted by $O\hat{x}\hat{y}\hat{z}$.

$$\begin{aligned} \vec{\hat{X}} &= \mathbf{B}\vec{\tilde{X}} \\ &= \begin{pmatrix} \cos\xi_5 & 0 & -\sin\xi_5 \\ 0 & 1 & 0 \\ \sin\xi_5 & 0 & \cos\xi_5 \end{pmatrix} \vec{\tilde{X}} \end{aligned} \tag{A.20}$$

Finally, the third rotation, through the angle ξ_6 about the $\hat{z}$ axis, brings the axes into coincidence with the $Ox'y'z'$ axes:

$$\begin{aligned} \vec{\hat{X}}' &= \mathbf{C}\vec{\hat{X}} \\ &= \begin{pmatrix} \cos\xi_6 & \sin\xi_6 & 0 \\ -\sin\xi_6 & \cos\xi_6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{\hat{X}} \end{aligned} \tag{A.21}$$

The complete transformation is obtained by applying these in order, according to the usual rules of matrix multiplication:

$$\mathbf{D} = \begin{pmatrix} c_5c_6 & c_4s_6 + s_4s_5c_6 & s_4s_6 - c_4s_5c_6 \\ -c_5s_6 & c_4c_6 + s_4c_5s_6 & s_4c_6 + c_4s_4s_6 \\ s_5 & -s_4c_5 & c_4c_5 \end{pmatrix} \tag{A.22}$$

where $s_n = \sin\xi_n, c_n = \cos\xi_n, n = 4, 5, 6$ and $\mathbf{D}^{-1}$ is equal to the transpose matrix of $\mathbf{D}$.

If $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ and $\vec{\alpha} = (\xi_4, \xi_5, \xi_6)$ can be expanded in the following form;

$$\vec{\xi} = \epsilon\vec{\xi}^{(1)} + \epsilon^2\vec{\xi}^{(2)} + O(\epsilon^3) \tag{A.23}$$

$$\vec{\alpha} = \epsilon\vec{\alpha}^{(1)} + \epsilon^2\vec{\alpha}^{(2)} + O(\epsilon^3) \tag{A.24}$$

then this becomes:

$$\begin{aligned} \mathbf{D} &= \mathbf{D}^{(0)} + \epsilon\mathbf{D}^{(1)} + \epsilon^2\mathbf{D}^{(2)} + O(\epsilon^3) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \xi_6 & -\xi_5 \\ -\xi_6 & 0 & \xi_4 \\ \xi_5 & -\xi_4 & 0 \end{pmatrix} \\ &\quad -\frac{1}{2}\begin{pmatrix} \xi_5^2 + \xi_6^2 & -2\xi_4\xi_5 & -2\xi_4\xi_6 \\ 0 & \xi_4^2 + \xi_6^2 & -2\xi_5\xi_6 \\ 0 & 0 & \xi_4^2 + \xi_5^2 \end{pmatrix} + O(\epsilon^3) \end{aligned} \tag{A.25}$$

Thus, from Eq.(A.18) $\vec{X}$ can be expressed as:

$$\vec{X} = \vec{X}' + \epsilon[\vec{\xi}^{(1)} + \vec{\alpha}^{(1)} \times \vec{X}'] + \epsilon^2[\vec{\xi}^{(2)} + \vec{\alpha}^{(2)} \times \vec{X}'] + \epsilon^2\mathbf{H}\vec{X}' + O(\epsilon^3) \tag{A.26}$$

where

$$\epsilon^2\mathbf{H} = \epsilon^2[\mathbf{D}^{(2)}]^{-1} \tag{A.27}$$

If X' is a fixed point vector, the velocity is

$$\vec{u} = \dot{\vec{X}} = \epsilon[\dot{\vec{\xi}}^{(1)} + \dot{\vec{\alpha}}^{(1)} \times \vec{X}'] + \epsilon^2[\dot{\vec{\xi}}^{(2)} + \dot{\vec{\alpha}}^{(2)} \times \vec{X}'] + \epsilon^2\dot{\mathbf{H}}\vec{X}' + O(\epsilon^3) \tag{A.28}$$

where the dot • denotes time derivative.

Let $\vec{n}$ be a unit vector normal to the body, directed into the body. In the $O'x'y'z'$ axes, the same vector is denoted by $\vec{n}'$. Since $\vec{n}$ does not depend on the translation vector $\vec{\xi}$, from (A.19) to (A.24) it is represented by

$$\vec{n} = \vec{n}' + \epsilon[\vec{\alpha}^{(1)} \times \vec{n}'] + \epsilon^2[\vec{\alpha}^{(2)} \times \vec{n}'] + \epsilon^2\mathbf{H}\vec{n}' + O(\epsilon^3) \tag{A.29}$$

while a rotational normal vector is obtained from vector products of (A.26) and (A.29) as:

$$\vec{X}\times\vec{n} = \vec{X}'\times\vec{n}' + \vec{\xi}\times\vec{n}' + \vec{\alpha}\times(\vec{X}'\times\vec{n}') + \vec{\xi}\times(\vec{\alpha}\times\vec{n}') + \epsilon^2\mathbf{H}(\vec{X}'\times\vec{n}') + O(\epsilon^3) \tag{A.30}$$

since the following vector relation holds:

$$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = \vec{0}$$

Next we shall consider a kinematic condition on the body itself. We assume that the body surface can be described in $O'x'y'z'$ axes (the body fixed coordinate system) by an equation of the form

$$S'(x', y', z') = 0 \tag{A.31}$$

or in $Oxyz$ axes (space fixed coordinate system)

$$S(x, y, z, t) = 0 \tag{A.32}$$

The hydrodynamic problem forces us to use the $Oxyz$ coordinates, in which the unit normal on $S = 0$ is a function of time. So we reinterpret (A.31):

> $S'(x, y, z) = 0$ specified the body surface in its equilibrium position, and we use the transformation of coordinates to express $\vec{n}$ in terms of $\vec{n}'$, the unit normal vector to the body surface in its mean position.

As noted earlier,(A.29) provides this relationship. In order to avoid any possible ambiguity, we shall use the following notations:

S : exact wetted surface, described with respect to $Oxyz$ axes ($S(x, y, z, t) = 0$);

S' : exact wetted surface, described with respect to $O'x'y'z'$ axes ($S'(x', y', z') = 0$);

S_m : wetted surface of the body in its equilibrium condition ($S'(x, y, z) = 0$).

The boundary condition is

$$\vec{n}\nabla\Phi = \vec{n} \bullet \vec{u} \quad \text{on} S \tag{A.33}$$

where $\vec{u}$ is the velocity of the surface S. Equation (A.28) gives $\vec{u}$ in terms of the vector $\vec{X}'$ of a body point in the body fixed coordinate system, but we reinterpret $\vec{X}'$ as the position vector in the $Oxyz$ axes of that body point when the body is at rest. So, if we replace $\vec{X}'$ by $\vec{X}$ in (A.28) and consider $\vec{X}$ as a point on the surface S_m, then (A.28) gives the actual velocity of that point on the body, but referred to the location of the point on S_m. Similarly, we use (A.29) to give the actual normal vector, but referred to the corresponding point on S_m.

In (A.33) $\nabla\Phi$ has to be evaluated. We assume that the velocity potential and its derivatives can be evaluated on the exact surface through Taylor expansions with respect to points on the mean surface:

$$\nabla\Phi = \epsilon\nabla\varphi_1 + \epsilon^2\nabla\varphi_2 + O(\epsilon^3) \tag{A.34}$$

and

$$\nabla\varphi_i = \nabla\varphi_i^m + [(\vec{X} - \vec{X}')\nabla]\nabla\varphi_i^m + \cdots \text{ on } S_m \tag{A.35}$$

where $(\vec{X} - \vec{X}')$ is given by (A.26).

Substituting (A.35) into (A.34), and using Eqs.(A.23) and (A.24), Eq.(A.34) can be expanded by the quantities of the body surface S_m as:

$$\nabla\Phi = \epsilon\nabla\varphi_1^m + \epsilon^2\{\nabla\varphi_2^m + [(\vec{\xi}^{(1)} + \vec{\alpha}^{(1)} \times \vec{X}')\nabla]\nabla\varphi_1^m\} + O(\epsilon^3) \tag{A.36}$$

Now, when all of the terms are organized according to powers of ϵ, we have the pairs of conditions:

$$O(\epsilon) \quad : \vec{n}'\nabla\varphi_1^m = \vec{n}'[\vec{\xi}^{(1)} + \vec{\alpha}^{(1)} \times \vec{X}'] \quad \text{on } S_m \tag{A.37}$$

$$O(\epsilon^2) \quad : \vec{n}'\nabla\varphi_2^m = \vec{n}'[\vec{\xi}^{(2)} + \vec{\alpha}^{(2)} \times \vec{X}'] + \dot{\mathbf{H}}\vec{X}' - [(\vec{\xi}^{(1)} + \vec{\alpha}^{(1)} \times \vec{X}')\nabla]\nabla\varphi_1^m$$
$$+(\vec{\alpha}^{(1)} \times \vec{n}')[(\vec{\xi}^{(1)} + \vec{\alpha}^{(1)} \times \vec{X}') - \nabla\varphi_1^m] \tag{A.38}$$

where all quantities on the right hand sides are to be evaluated on S_m.

Condition (A.37) is familiar from ship motion theory. In (A.38), the left hand side and the first term on the right hand side are identical to (A.37) except that the index 1 is replaced by 2. With respect to the other terms on the right hand side, it is observed that the second term accounts for nonlinear effects included in the velocity $\vec{u}$, the third term corrects for the fact that $\vec{n}'\nabla\varphi_1^m$ in (A.37) is figured on the mean position of the body instead of on its instantaneous position, and that the last term accounts for the difference in direction between $\vec{n}$ and $\vec{n}'$.

A.4 Force and moment

The three components of force and three components of moment on the body can also be expressed initially as follows:

$$F_i(t) = \iint_S n_i p dS \qquad (i = 1, .., 6) \tag{A.39}$$

where S is the exact wetted surface of the body, and $p = p(x, y, z, t)$ is the fluid pressure on the body surface. The six quantities n_i, defined in (A.29) and (A.30), must be evaluated instantaneously as functions of time.

In order to proceed further analytically, we transform the integral over S into an integral over S_m, the wetted surface of the body in its equilibrium position in calm water. This requires two kinds of adjustments.

The first is that since S is displaced and rotated with respect to S_m, we must express p and n_i in terms of their values on S_m . For the latter, we use (A.29) and (A.30), the primes now denoting quantities evaluated on S_m. For

p, we assume that values on S can be obtained in terms of a Taylor expansion with respect to S_m,

$$p\mid_S = p\mid_{S_m} + (\vec{X} - \vec{X}')\nabla p)\mid_{S_m} + \cdots \tag{A.40}$$

Since p contains a hydrostatic pressure term, $-\rho g z$, the first term on the right hand side is $O(1)$ and the second term is $O(\epsilon)$. But the hydrostatic pressure has no effect after the second term of the expansion, and so the first unwritten term is actually $O(\epsilon^3)$.

The other is that the integration over S is to be carried right up to the water surface, $z = \zeta$, but the integration over S_m goes up only to $z = 0$ which is equivalent to $z = \xi_3 + y\xi_4 - x\xi_5$ on S (as shown in Fig.A-2). Let ΔS be the part of S between $z = \xi_3 + y\xi_4 - x\xi_5$ and $z = \zeta$. Then, to second order,

$$\begin{aligned}\iint_{\Delta S} n_i p dS \quad &= -\rho \int_{C_m} dl \int_0^{\epsilon[\zeta - \xi_{31} - y\xi_{41} + x\xi_{51}] + \cdots} dz \\ &\times (n_i' + \cdots)\{gz + \epsilon[\varphi_{1t}^m + g(\xi_{31} + y\xi_{41} - x\xi_{51})] + \cdots\}\end{aligned} \tag{A.41}$$

where C_m is the intersection of S_m and the plane $z = 0$, and in the double indices, i.e. ξ_{ij}, the first, i, denotes the orientation of the axis, and the second, j, shows the term in the perturbation expansions. On the right hand side, we can now drop the prime on n_i', since the indicated domain of integration makes it clear that n_i is being evaluated on the mean position of the body. Two further simplifications can be made consistently:

$$\begin{aligned}&n_i(x, y, z) = n_i(x, y, 0) + O(\epsilon) \\ &\varphi_{1t}^m(x, y, z, t) = \varphi_{1t}^m(x, y, 0, t) + O(\epsilon) = -g\zeta_1(x, y, t)\end{aligned}$$

So Eq.(A.41) can be evaluated as:

$$\iint_{\Delta S} n_i p dS = -\frac{\rho g}{2}\epsilon^2 \oint_{C_m} dl n_i [\zeta_1 - \xi_{31} - y\xi_{41} + x\xi_{51}]^2 \tag{A.42}$$

Now let us consider the force. We have divided S into two parts, i.e. the main integral over S_m and the integral given by (A.41). Organizing the results by order of magnitude, we obtain

$$\begin{aligned}\vec{F} \quad &= -\rho g V \vec{k} \\ &-\epsilon\rho\{\iint_{S_m} \vec{n}\varphi_{1t}^m dS + gA_{WP}(\xi_{31} + y_f\xi_{41} - x_f\xi_{51})\vec{k}\} \\ &-\epsilon^2\rho \iint_{S_m} \{\vec{n}[\varphi_{2t}^m + \frac{|\nabla\varphi_1^m|^2}{2} + (\vec{\xi}^{(1)} + \vec{\alpha}^{(1)} \times \vec{X})\nabla\varphi_{1t}^m] + [\vec{\alpha}^{(1)} \times \vec{n}]\varphi_{1t}^m\} dS\end{aligned}$$

$$-\frac{g}{2}\epsilon^2 \oint_{C_m} dl\vec{n}[\zeta_1^2 - 2\zeta_1(\xi_{31} + y\xi_{41} - x\xi_{51})]$$

$$+g\epsilon^2 A_{WP}\{(\xi_{32} + y_f\xi_{42} - x_f\xi_{51}) + \xi_{61}(x_f\xi_{41} + y_f\xi_{51})\}\vec{k} + O(\epsilon^3) \quad \text{(A.43)}$$

where

$\vec{k}$ = unit vector in z direction

$\vec{n}, \vec{X}$ = normal and point vectors in the body-fixed coordinate system

V = volume of displaced water at equilibrium

$$V = \iint_{S_m} zdxdy = \iint_{S_m} xdydz = \iint_{S_m} ydxdz$$

A_{WP} = area of water plane at equilibrium

x_f = position of longitudinal centre of flotation

y_f = position of transverse centre of flotation

$$x_f A_{WP} = \iint_{S_m} xdxdy = \frac{1}{2}\oint x^2 dy$$
$$y_f A_{WP} = \iint_{S_m} ydxdy = \frac{1}{2}\oint y^2 dx$$

Next we divide the first and second order potentials into three parts in the following forms:

$$\begin{aligned} \varphi_1^m &= \varphi_1^I + \varphi_1^d + \varphi_1^r \\ \varphi_2^m &= \varphi_2^I + \varphi_2^d + \varphi_2^r \end{aligned} \quad \text{(A.44)}$$

where the indices I, d, and r denotes the incident wave potential, diffraction potential, and radiation potential respectively. Furthermore, we decompose Eq.(A.43) as:

$$\vec{F} = -\vec{F}_{HS}^{(0)} - \epsilon(\vec{F}_W^{(1)} + \vec{F}_{HD}^{(1)} + \vec{F}_{HS}^{(1)}) - \epsilon^2(\vec{F}_W^{(2)} + \vec{F}_{HD}^{(2)} + \vec{F}_{HS}^{(2)}) + O(\epsilon^3) \quad \text{(A.45)}$$

where the indices, W, HD, and HS denote the wave force, the hydrostatic force, and the hydrodynamic force respectively. Then, organizing the results by order of magnitude, we get:

$$O(1): \quad \vec{F}_{HS}^{(0)} = \rho g V \vec{k} \quad \text{(A.46)}$$

$$O(\epsilon): \quad \vec{F}_W^{(1)} = \rho \iint_{S_m} \vec{n}(\varphi_{1t}^I + \varphi_{1t}^d)dS \quad \text{(A.47)}$$

$$\vec{F}^{(1)}_{HD} = \rho \iint_{S_m} \vec{n}\varphi^{r}_{1t} dS \tag{A.48}$$

$$\vec{F}^{(1)}_{HS} = \rho g A_{WP}(\xi_{31} + y_f\xi_{41} - x_f\xi_{51})\vec{k} \tag{A.49}$$

$$O(\epsilon^2): \quad \vec{F}^{(2)}_{W} = -\frac{\rho g}{2}\oint_{C_m} dl\vec{n}[\zeta_1 - (\xi_{31} + y\xi_{41} - x\xi_{51})]^2$$
$$+\rho \iint_{S_m} \{\vec{n}[\varphi^{I}_{2t} + \varphi^{d}_{2t} + \frac{|\nabla\varphi^{m}_{1}|^2}{2} + (\vec{\xi}^{(1)} + \vec{\alpha}^{(1)} \times \vec{X})\nabla\varphi^{m}_{1t}]\}dS$$
$$+\vec{\alpha}^{(1)} \times \vec{F}^{(1)} + \rho g A_{WP}\xi_{61}(x_f\xi_{41} + y_f\xi_{51})\vec{k} \tag{A.50}$$

$$\vec{F}^{(2)}_{HD} = \rho \iint_{S_m} \vec{n}\varphi^{r}_{2t} dS \tag{A.51}$$

$$\vec{F}^{(2)}_{HS} = \rho g A_{WP}(\xi_{32} + y_f\xi_{42} - x_f\xi_{52})\vec{k} \tag{A.52}$$

where in order to lead (A.50) the following relation is used.

$$\iint_{S_m} (\vec{\alpha}^{(1)} \times \vec{n})\varphi_{1t} dS - \frac{g}{2}\oint_{C_m} dl\vec{n}[\zeta_1^2 - 2\zeta_1(\xi_{31} + y\xi_{41} - x\xi_{51})]$$
$$= \vec{\alpha}^{(1)} \times \vec{F}^{(1)} - \frac{g}{2}\oint_{C_m} dl\vec{n}[\zeta_1 - (\xi_{31} + y\xi_{41} - x\xi_{51})]^2 \tag{A.53}$$

where

$$\vec{F}^{(1)} = \vec{F}^{(1)}_{W} + \vec{F}^{(1)}_{HD} + \vec{F}^{(1)}_{HS} \tag{A.54}$$

From (A.50) it is found that the second order force, i.e. $F^{(2)}_{W}$, consists of the following five terms:

(1) The first term is the component caused by fluid pressure between mean and instantaneous wave surfaces

$$\vec{F}^{(2)}_{1} = -\frac{\rho g}{2}\oint_{C_m} dl\vec{n}[\zeta_1 - (\xi_{31} + y\xi_{41} - x\xi_{51})]^2 \tag{A.55}$$

(2) The second term comes from the quadratic pressure term in the Bernoulli equation.

$$\vec{F}^{(2)}_{2} = \frac{\rho}{2}\iint_{S_m} \vec{n}|\nabla\varphi^{m}_{1}|^2 dS \tag{A.56}$$

(3) The third arises from the variation of the acting point of fluid pressure.

$$\vec{F}_3^{(2)} = \rho \iint_{S_m} \vec{n}\{(\vec{\xi}^{(1)} + \vec{\alpha}^{(1)} \times \vec{X})\nabla\varphi_{1t}^m\}dS \tag{A.57}$$

(4) The fourth comes from the variation of direction of first order wave force with respect to rotation of a body.

$$\vec{F}_4^{(2)} = \vec{\alpha}^{(1)} \times \vec{F}^{(1)} \tag{A.58}$$

(5) The last term is the component due to second order potentials

$$\vec{F}_5^{(2)} = \rho \iint_{S_m} \vec{n}(\varphi_{2t}^I + \varphi_{2t}^d)dS \tag{A.59}$$

Appendix B

Estimation of Cross Bispectrum

FFT, BT and MEM method have been used as the estimation of auto and cross spectra. But the general estimation of bispectra has not been developed so far. In this section, we shall introduce the Dalzell's method[2)] as one example of the estimations of cross bispectrum.

Since the sample(one record obtained by experiments) is necessarily finite, it is possible only to estimate cross bispectral averages rather than actual densities:

$$\breve{C}(\Omega_1, \Omega_2) = \int\int H(\Omega_3, \Omega_4) C(\Omega_1 + \Omega_3, \Omega_2 + \Omega_4) d\Omega_3 d\Omega_4 \qquad \text{(B.1)}$$

where the average $H(\Omega_3, \Omega_4)$ is weighted by the kernel function, which is called "*cross bispectral window*". The window by analogy with scalar spectrum analysis must take a peak at a bi-frequency $(0.0, 0.0)$, fall off rapidly elsewhere, and remain near zero away from the peak. As for a usual scalar spectrum analysis, a too-broad window makes the estimates bad and a too- narrow window with respect to sample length increases the variance of the estimate. It is clear that since the window is for averaging over frequency, its integral should be unity;

$$\int\int H(\Omega_3, \Omega_4) d\Omega_3 d\Omega_4 = 1 \qquad \text{(B.2)}$$

Because the data is sequentially sampled at time interval Δt, bi-frequencies outside the principal range:

$$-\tfrac{\pi}{\Delta t} < \Omega_1 < \tfrac{\pi}{\Delta t}$$
$$-\tfrac{\pi}{\Delta t} < \Omega_2 < \tfrac{\pi}{\Delta t}$$

are aliased with those inside. It is assumed that the data is sampled at a sufficiently short time interval so that the cross bispectrum is negligible outside the principal range. According to Dalzell's work[2], the time interval should be about half the interval for a scalar spectrum analysis. Because the data is sampled sequentially, a lag window of the form:

$$h(\tau_1,\tau_2)=\sum_j\sum_k a_j b_k \delta(\tau_1-j\Delta t)\delta(\tau_2-k\Delta t) \tag{B.3}$$

is chosen, where the a_j and b_k are real, and $\delta(t)$ is the Dirac's delta function. Then the cross bispectrum estimate is given by:

$$\breve{C}(\Omega_1,\Omega_2)=\sum_{j=-m}^{m}\sum_{k=-n}^{n}\tilde{R}_{\eta\eta X}(-j\Delta t,-k\Delta t)a_j b_k \exp\{i\Delta t(j\Omega_1+k\Omega_2)\} \tag{B.4}$$

This estimator involves the third order correlation function. Setting time, that is, $t = n\Delta t$, the correlation function can be expressed by the form readily available with the sequentially sampled data as:

$$\tilde{R}_{\eta\eta X}(-j\Delta t,-k\Delta t)=E[\tilde{\eta}(n\Delta t+j\Delta t)\tilde{\eta}(n\Delta t-j\Delta t)\{X(n\Delta t+k\Delta t)-\overline{X}\}] \tag{B.5}$$

The expected value is conventionally estimated by a summation over the available sample divided by the sample length and this interpretation is followed so that

$$\begin{aligned}\breve{R}(j,k) &= \frac{1}{N'}\sum_n \eta'(n+j)\eta'(n-j)X'(n+k)\\ &\equiv \tilde{R}_{\eta\eta X}(-j\Delta t,-k\Delta t)\end{aligned} \tag{B.6}$$

where:

N' = number of products summed

$\eta'(n)$ = wave elevation time series corrected to zero sample mean

$X'(n)$ = nonlinear response time series corrected to zero sample mean

Next, the main problem is to construct the cross bispectral window $H(\Omega_3,\Omega_4)$. Considering m and n as maximum lags, Eq.(B.4) becomes:

$$\breve{C}(\Omega_1,\Omega_2)=\sum_{j=-m}^{m}\sum_{k=-n}^{n}\breve{R}(j,k)a_j b_k \exp\{i\pi(j\frac{l_1}{m}+k\frac{l_2}{n}\} \tag{B.7}$$

where $\Omega_1=\frac{\pi l_1}{m\Delta t}, \Omega_2=\frac{\pi l_2}{n\Delta t}$.

In choosing the cross bispectral window, it was assumed that the natural choice would be a two-dimensional analogy with the spectral windows to be used in the estimation of the wave spectrum. The continuous lag window corresponding to the scalar spectrum window is of the form:

$$A(\tau) = q[e_1 + e_2 \cos(\frac{\pi\tau}{m\Delta t})] \tag{B.8}$$

where e_1 and e_2 are coefficients of window function.

Since optimum window function for cross bispectrum estimate has not been known yet, we determined the coefficients as $e_1 = 0.54$ and $e_2 = 0.46$. These coefficients are equal to ones of the Hamming type window function. In Eq.(B.3) the lag window is a product, one factor for each lag variable. The factors may be different. This would result in spectral windows differently shaped in the sum and difference frequencies, as well as having different bandwidths. There appeared no justification for a difference in normalized shape of the window. It was assumed that a discrete version of the scalar spectrum lag window would be appropriate for each of the factors, so that in Eq.(B.3) let:

$$\begin{aligned} a_j(\tau) &= q_j[e_1 + e_2 \cos(\tfrac{\pi j}{m})] \\ b_k(\tau) &= q_k[e_1 + e_2 \cos(\tfrac{\pi k}{n})] \end{aligned} \tag{B.9}$$

where q_j and q_k are constants independent j and k and they are determined by normalization condition, that is, Eq.(B.2). Since the cross bispectral window $H(\Omega_3, \Omega_4)$ and the lag window are a transform pair and thus:

$$\begin{aligned} H(\Omega_3, \Omega_4) &= \frac{q_j q_k}{(2\pi)^3}[\sum_{j=-m}^{m} (e_1 + e_2 \cos(\frac{\pi j}{m}))\cos(\Delta t j \Omega_3)] \\ &\times [\sum_{k=-n}^{n} (e_1 + e_2 \cos(\frac{\pi k}{n}))\cos(\Delta t k \Omega_4)] \end{aligned} \tag{B.10}$$

This result shows that the cross bispectral window is real and symmetric in j and k, and is continuous in Ω_1 and Ω_2. The window also has a period of $\Omega = \frac{2\pi}{\Delta t}$.

Unknown constants q_j and q_k can be determined from normalization condition, Eq.(B.2) as:

$$q_j q_k = \{\frac{\Delta t}{(e_1 + e_2)}\}^2 \tag{B.11}$$

Then the estimate of the cross bispectral averages becomes:

$$\begin{aligned} \check{C}(\Omega_1, \Omega_2) &= \{\frac{\Delta t}{2\pi(e_1 + e_2)}\}^2 \sum_{j=-m}^{m} \sum_{k=-n}^{n} (e_1 + e_2 \cos(\frac{\pi j}{m}))(e_1 + e_2 \cos(\frac{\pi k}{n})) \\ &\times \exp\{i\pi(\frac{l_1 j}{m} + \frac{l_2 k}{n})\} \frac{1}{N'} \sum_{n} \eta'(n+j)\eta'(n-j)X'(n+k) \end{aligned} \tag{B.12}$$

Multiplying $\Delta\Omega_1 \bullet \Delta\Omega_2$ in the above equation and summing over all values of l_1 and l_2, the integration of the cross bispectrum approaches the following form:

$$\Delta\Omega_1 \Delta\Omega_2 \sum_{l_1,l_2} \breve{C}(\Omega_1,\Omega_2) = 4\pi^2 \breve{R}(0,0)[1 + \frac{1}{2m} + \frac{1}{2n} + \frac{1}{4nm} + O(\frac{1}{(nm)^2})] \tag{B.13}$$

Thus the estimate of the cross bispectrum has a error. But the error is small compared with the true value and is negligible for practical values of m and n.

Appendix C

Viscous drift force acting on a vertical circular cylinder with small diameter

The forces on a small vertical cylinder due to waves is represented by the Morison equation. For a unit length of the submerged portion of the cylinder, the force is given by:

$$f_x = \frac{C_m \rho \pi D^2}{4}\dot{u} + \frac{\rho D C_d}{2} u|u| \tag{C.1}$$

where u is a horizontal component of wave particle velocities, D is a diameter of the cylinder, and C_m and C_d are inertia and drag coefficients.

If current is not included and the linear wave theory can be applied, a surface elevation $\zeta(t)$ and the horizontal component of wave particle velocities are represented in the following form:

$$\zeta(t) = \frac{H_w}{2}\cos\omega t \tag{C.2}$$

$$u(t) = \frac{H_w \omega}{2}\exp(\kappa z)\cos\omega t \tag{C.3}$$

where H_w is a wave height, κ is a wave number, and ω is a circular wave frequency.

Substituting the equations (C.2) and (C.3) into (C.1), the horizontal force acting on the vertical cylinder can be expressed as:

$$F_x = \int_{-h}^{\zeta} f_x dz$$

$$= [-\frac{Cm\pi\rho D^2 Hw\kappa g}{8}\sin\omega t + \frac{C_d\rho DH_w^2\omega^2}{8}\cos\omega t|\cos\omega t|]$$
$$\times\{\frac{\exp(\kappa\zeta)-\exp(-\kappa h)}{\kappa}\} \quad \text{(C.4)}$$

Thus if it is assumed that $\kappa\zeta \ll 1$, F_x can be divided into the following two parts($F_x^{(1)}$ and $F_x^{(2)}$):

$$F_x^{(1)} = [-\frac{C_m\pi\rho D^2 H_w\kappa g}{8}\sin\omega t + \frac{C_d\rho DH_w^2\omega^2}{8}\cos\omega t|\cos\omega t|]$$
$$\times\{\frac{1-\exp(-\kappa h)}{\kappa}\} \quad \text{(C.5)}$$

$$F_x^{(2)} = [-\frac{C_m\pi\rho D^2 H_w\kappa g}{8}\sin\omega t + \frac{C_d\rho DH_w^2\omega^2}{8}\cos\omega t|\cos\omega t|]\zeta \quad \text{(C.6)}$$

where $\frac{F_x^{(1)}}{D^2}$ is the force per section area integrated over $-h$ to 0.0 with respect to z. It expresses a first order force when $\kappa\zeta = O(\epsilon)$. And $F_x^{(2)}$ indicates a higher order force and it does not depend on the draft. In F_x, the most important term for the drift force is $F_x^{(2)}$, which can also be represented by the alternative form like:

$$F_x^{(2)} = f_x\,|_{z=0} \times\zeta \quad \text{(C.7)}$$

Namely this is the product of the wave elevation ζ and the horizontal force per unit length at the still water surface, that is, it is the wave force integrated over the range from the still water surface to the instantaneous wave surface when the horizontal wave particle velocity is distributed as shown by Fig.(C-1). Thus the force expressed by Eq.(C.7) is called a "**free surface force**".

Since the linear wave theory is applicable only for infinitesimal wave amplitudes and it is valid up to the still water level, extension of expression for the water particle kinematics up to the free surface of a finite amplitude wave is questionable. Therefore in order to exactly discuss, it is necessary to use the finite wave amplitude theory. But since our interest is to study fundamental characteristics of a viscous drift force, we dare to use the linear wave theory.

By using the Hilbert transform, an out-of-phase component of the surface elevation ζ can be expressed by:

$$\eta(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\zeta(\tau)}{(t-\tau)}d\tau \quad \text{(C.8)}$$

Then the horizontal velocity component u_0 on $z = 0$ is given by the derivative of η as follows:

$$u_0 = \dot{\eta}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\zeta(\tau)}{(t-\tau)^2}d\tau \quad \text{(C.9)}$$

And similarly the horizontal acceleration becomes:

$$\dot{u}_0 = \ddot{\eta}(t) = -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\zeta(\tau)}{(t-\tau)^3}d\tau \quad \text{(C.10)}$$

From the characteristics of the Hilbert transform, it is easily found that u_0 is in phase with ζ, and $\dot{u}_0$ out of phase with it.

It is well known that C_d and C_m in the Morison equation are the functions of the Keulegan-Carpenter number and the Reynolds number. In addition, we may assume that these hydrodynamic force coefficients can also be represented by a function of wave frequency.

$$f_x(t) = \int_T g_1(\tau)\dot{u}_0(t-\tau)d\tau + \int_T g_2(\tau)u_0(t-\tau)|u_0(t-\tau)|d\tau \tag{C.11}$$

Similarly applying this system representation for Eq.(C.7), $F_x^{(2)}$ has:

$$\begin{aligned} F_x^{(2)}(t) &= \int_T g_1(\tau)\dot{u}_0(t-\tau)\zeta(t-\tau)d\tau \\ &+ \int_T g_2(\tau)u_0(t-\tau)|u_0(t-\tau)|\zeta(t-\tau)d\tau \end{aligned} \tag{C.12}$$

Now, by using the equivalent linearization technique, $u_0|u_0|$ can be approximated in the following forms:

i) in the case that ζ is a regular wave process,

$$u_0|u_0| = \alpha u_0 = \frac{4\pi H_w \omega}{3} u_0 \tag{C.13}$$

ii) in the case that ζ is a Gaussian random process,

$$u_0|u_0| = \alpha u_0 = \sqrt{\frac{8}{\pi}}\sigma_{u_0} u_0 \tag{C.14}$$

where σ_{u_0} is the standard deviation of u_0.

Since α is a function of wave frequency, it can be included in the system function h_2. Thus Eq.(C.12) can be rewritten as:

$$F_x^{(2)}(t) = \int_T g_1(\tau)\dot{u}_0(t-\tau)\zeta(t-\tau)d\tau + \int_T h_2(\tau)u_0(t-\tau)\zeta(t-\tau)d\tau \tag{C.15}$$

From the relationship between u_0 and ζ, the second term on right hand side of the above equation includes a slowly varying drift force but the first term does not. Hereafter we shall consider only the second term in Eq.(C.15). From Eq.(C.9),then, the second term can be represented in the following form:

$$F_{xs}^{(2)}(t) = \frac{1}{\pi}\int_{T_1}\int_{T_2} \frac{h_2(\tau_1)}{(\tau_2-\tau_1)^2}\zeta(t-\tau_1)\zeta(t-\tau_2)d\tau_1 d\tau_2 \tag{C.16}$$

If we define the new function g_2 by

$$g_2(\tau_1,\tau_2) = \frac{1}{2\pi}\left[\frac{h_2(\tau_1)}{(\tau_2+\tau_1)^2} + \frac{h_2(\tau_2)}{(\tau_2+\tau_1)^2}\right] \tag{C.17}$$

$F_{xs}^{(2)}$ indicates the second term in Volterra functional series, that is, g_2 is equivalent to the quadratic impulse response function. Using the Fourier transform of generalized function[3] as

$$\int_{-\infty}^{\infty} x^{-m} e^{-ixy} dx = -\frac{\pi i(-iy)^{m-1}}{(m-1)!} sgn(y), \qquad (C.18)$$

the Fourier transform of Eq.(C.17) becomes:

$$G_2(\omega_1, \omega_2) = \frac{1}{4}[|\omega_2| Q(\omega_1 - \omega_2) + |\omega_1| Q(\omega_1 - \omega_2)] \qquad (C.19)$$

where Q is the Fourier transform of h_2.

Finally, the quadratic transfer function of slowly varying drift force due to viscous effect can be represented by:

$$G_2(\omega_1, -\omega_2) = \frac{1}{4}[|\omega_2| Q(\omega_1 + \omega_2) + |\omega_1| Q(\omega_1 + \omega_2)] \qquad (C.20)$$

If the drag coefficients C_d does not depend on the wave frequency and the waves are the combined regular waves with two frequencies, the quadratic transfer function G_2 is proportional to the square of mean frequency of two wave components. And if the wave system consists of a single-frequency wave, G_2 becomes:

$$G_2(\omega, -\omega)(\frac{H_w}{2})^2 = \frac{H_w^3}{12} C_d \rho \pi D \omega^2 \qquad (C.21)$$

This result agree with the result obtained by Standing and the others[4], that is, the viscous steady drift force is proportional to the wave amplitude to the third power.

Appendix D

On the effect of exciting short period disturbances on the free and forced oscillations for the system with nonlinear damping

Free oscillation tests have been used for measuring the damping coefficient of a ship or a floating offshore structure. Especially, since moored offshore structures have a long natural period in surge motion in general and the damping force is very small at this period, the experiment is one of the best ways to get the damping force.

This appendix shows the analytical results on the influence of exciting high frequency disturbances on free and forced oscillations for the system, the damping force of which is assumed proportional to the square of velocity, and it concludes that the damping force coefficients increase by the exciting high frequency disturbances.

D.1 Free oscillation in regular high frequency exciting disturbance

The free oscillation equation including exciting disturbance $E(t)$ is described in the following equation:

$$M\ddot{X} + N\dot{X}|\dot{X}| + KX = E(t) \tag{D.1}$$

where M is the total mass coefficient, N is the damping force coefficient, K the spring constant, and the dots represent the derivatives with respect to time.

Considering Eq.(D.1) in the time when X becomes a negative value, replacing E by $\beta M\cos(n_e t)$ and dividing the both side of Eq.(D.1) by M, Eq.(D.1) becomes as follows:

$$\ddot{X} - \alpha\dot{X}^2 + n^2 X = \beta\cos(n_e t) \tag{D.2}$$

where $\alpha = \frac{N}{M}, n^2 = \frac{K}{M}$, and n and n_e are unequal.

If α is small, the solution of Eq.(D.2) and n^2 can be expanded by α. Namely, X and n^2 are expressed in the following form:

$$\begin{aligned} X &= X_0 + \alpha X_1 + \alpha^2 X_2 + \cdots \\ n^2 &= n_0^2 + \alpha n_1^2 + \alpha^2 n_2^2 + \cdots \end{aligned} \tag{D.3}$$

Substituting Eq.(D.3) into Eq.(D.2) and ordering Eq.(D.2) in term α.

$$O(1): \quad \ddot{X}_0 + n_0^2 X_0 = \beta\cos(n_e t) \tag{D.4}$$

$$O(\alpha): \quad \ddot{X}_1 + n_0^2 X_1 = \dot{X}_0^2 - n_1^2 X_0 \tag{D.5}$$

$$O(\alpha^2): \quad \ddot{X}_2 + n_0^2 X_2 = -n_1^2 X_1 - n_2^2 X_0 + 2\dot{X}_0\dot{X}_1 \tag{D.6}$$

If the initial conditions of Eq.(D.2) are $\dot{X} = 0$ and $X = a$, the initial conditions corresponding to Eq.(D.4), (D.5), and (D.6) are as follows:

$$\begin{aligned} X_0 &= a, \dot{X}_0 = 0 \\ X_1 &= 0, \dot{X}_1 = 0 \\ X_2 &= 0, \dot{X}_2 = 0 \end{aligned} \tag{D.7}$$

Accordingly, if the resonance phenomena do not occur and the ratio between the natural frequency n_0 and the frequency of exciting disturbance n_e is large enough, the period of one cycle and the decaying ratio of amplitude a_n are obtained approximately by:

$$\frac{a_{n+1}}{a_n} = 1 + \alpha\left(\frac{4a}{3} + \frac{2\bar{\beta}^2}{3ak^2}\right) \tag{D.8}$$

where $\bar{\beta} = \frac{\beta}{n_0^2}$ and $k = \frac{n_e}{n_0}$. From Eq.(D.8) it is found that the exciting disturbance exerts an influence on the decaying ratio of amplitude and the period of one cycle. Thus, in the case of measuring the damping coefficient from the result of free oscillation test, we must use the amplitude which satisfies the following relation:

$$a_n \gg \frac{\bar{\beta}}{\sqrt{2}n_1 n_e} \tag{D.9}$$

Let us confirm the above result by the numerical calculation. As a free oscillation equation, we consider the following equation:

$$\ddot{X} + \dot{X}|\dot{X}| + 9X = \beta \sin(30t) \tag{D.10}$$

The numerical results calculated for both $\beta = 0$ and $\beta = 10$ by use of Runge-Kutta-Gill method when the initial conditions are $X = 0$ and $X = 1$ are shown in Fig.D.1. From this figure it is found that the decay of amplitude in exciting disturbance is larger than that in still water. Furthermore it is confirmed that the effect of exciting disturbance occurs within

$$a_n \leq 0.08 \tag{D.11}$$

calculated by Eq.(D.9).

D.2 Forced oscillation in exciting disturbance

When the regular and irregular exciting disturbances are added into the oscillation system with nonlinear damping force, the differential equation of oscillation can be expressed in the following form:

$$M\ddot{X} + f(\dot{X}) + KX = E_1(t) + E_2(t) \tag{D.12}$$

where M, N and K are the same coefficients in the previous section, $f(\dot{X})$ is the nonlinear damping force and E_1 and E_2 are the regular and irregular exciting disturbances, respectively.

If the nonlinearity of Eq.(D.12) is not so strong, it is considered that the response of Eq.(D.12) can be represented in sum of the linear responses due to E_1 and $E2$. Namely, if z_1 and z_2 are the linear velocity responses due to $E1$ and $E2$ respectively, the nonlinear velocity response may be expressed in the following representation:

$$f(z_1 + z_2) \sim \kappa_1 z_1 + \kappa_2 z_2 \tag{D.13}$$

So, we consider the following functional

$$J = E[(Nf(z_1 + z_2) - \kappa_1 z_1 - \kappa_2 z_2)^2] \tag{D.14}$$

and determine κ_1 and κ_2 such that minimize J, that is, we shall apply the so-called equivalent linearization method. Then the Equivalent Linearized Damping (E.L.D.) coefficients κ_1 and κ_2 are given as follows:

$$\begin{aligned} \kappa_1 &= \frac{E[z_1 f(z_1 + z_2)]}{E[z_1^2]} \\ &= \frac{1}{\sigma_1^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_1 f(z_1 + z_2) p_1(z_1) p_2(z_2) dz_1 dz_2 \end{aligned} \tag{D.15}$$

$$\begin{aligned}\kappa_2 &= \frac{E[z_2 f(z_1+z_2)]}{E[z_2^2]} \\ &= \frac{1}{\sigma_2^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_2 f(z_1+z_2)p_1(z_1)p_2(z_2)dz_1dz_2 \end{aligned} \tag{D.16}$$

where $E[\bullet]$ denotes the expectation, $p_i(i=1,2)$ are the probability density functions of $z_i(i=1,2)$ and $\sigma_i^2(i=1,2)$ are the mean square of $z_i(i=1,2)$. By use of the characteristic function ϕ_i for z_i and the Fourier transform F for $f(z_i)$, Eqs.(D.15) and (D.16) can be rewritten in the form:

$$\kappa_j = -\frac{i}{2\pi\sigma_j^2}\int_{-\infty}^{\infty} F(i\omega)\phi_k \frac{d\phi_j}{d\omega}d\omega \quad (j,k=1,2,j\neq k) \tag{D.17}$$

where

$$\phi_j(\omega) = \int_{-\infty}^{\infty} \exp(i\omega z_j)p_j(z_j)dz \qquad \text{for } j=1,2$$

$$F(i\omega) = \int_{-\infty}^{\infty} \exp(-i\omega\dot{x})f(\dot{x})d\dot{x}$$

Let us now apply for the oscillation system with nonlinear damping force $f(\dot{X}) = \dot{X}|\dot{X}|$. If z_1 is $A\sin(\omega t)$ and z_2 is the zero-mean stationary Gaussian process, the E.L.D. coefficients κ_1 and κ_2 are given by:

$$\kappa_1 = \frac{8}{\pi A}\int_0^{\infty} \frac{J_1(A\omega)}{\omega^3}\exp(-\frac{\sigma_2^2\omega^2}{2})d\omega \tag{D.18}$$

$$\kappa_2 = \frac{4}{\pi}\int_0^{\infty} \frac{J_0(A\omega)}{\omega^2}\exp(-\frac{\sigma_2^2\omega^2}{2})d\omega \tag{D.19}$$

where J_0 and J_1 are the Bessel functions of the fist kind.

Considering the following relation

$$\begin{aligned}\textstyle\int_0^{\infty} J_\nu(at)\exp(-b^2t^2)t^{\mu-1}dt &= \frac{\Gamma(\frac{\nu}{2}-\frac{\mu}{2}+1)}{2b^\mu\Gamma(\nu+1)}\exp(-\frac{a^2}{4b^2}) \\ &\times(\frac{a}{2b})^\nu {}_1F_1(\frac{\nu}{2}-\frac{\mu}{2}+1;\nu+1;\frac{a^2}{4b^2}) \end{aligned} \tag{D.20}$$

Eqs.(D.18) and (D.19) can be expressed in the form:

$$\kappa_1 = \sqrt{\frac{8}{\pi}}\sigma_2\exp(-\frac{A^2}{2\sigma_2^2}){}_1F_1(2.5;2;\frac{A^2}{2\sigma_2^2}) \tag{D.21}$$

$$\kappa_2 = \sqrt{\frac{8}{\pi}}\sigma_2\exp(-\frac{A^2}{2\sigma_2^2}){}_1F_1(1.5;1;\frac{A^2}{2\sigma_2^2}) \tag{D.22}$$

where Γ is the Gamma function and ${}_1F_1$ is the confluent hypergeometric function[5]. From Eqs.(D.21) and (D.22) it is found that the E.L.D. coefficients change with the energy ratio between z_1 and z_2.

When $\alpha > \gamma$, the asymptotic expansion of ${}_1F_1(\alpha;\gamma;z)$ for the large value of $|z|$ is as follows:

$$ {}_1F_1(\alpha;\gamma;z) \sim \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \exp(z) z^{\alpha-\gamma} \tag{D.23}$$

Accordingly, when $\frac{A^2}{2\sigma_2^2} \gg 1$,

$$\kappa_1 \sim \frac{8A}{3\pi} \tag{D.24}$$

$$\kappa_2 \sim \frac{4A}{\pi} \tag{D.25}$$

The result of Eq.(D.24) is identical with that of only E_1.

When $\frac{A^2}{2\sigma_2^2} \ll 1$,

$$\kappa_1 = \kappa_2 \sim \sqrt{\frac{8}{\pi}}\sigma_2 \tag{D.26}$$

This result coincides with the result for pure Gaussian input E_2. Figures D.2 and D.3 show the calculated results of κ_1 and κ_2. From these figures it is found that the interaction effect due to two exciting disturbances on the E.L.D. coefficients is large. The energy dissipation consumed by nonlinear damping can be expressed as follows:

$$\begin{aligned} \overline{E} &= \kappa_1 \frac{A^2}{2} + \kappa_2 \sigma_2^2 \\ &= \sqrt{\frac{8}{\pi}} \sigma_2^3 \exp(-\frac{A^2}{2\sigma_2^2}) {}_1F_1(2.5; 1; \frac{A^2}{2\sigma_2^2}) \end{aligned} \tag{D.27}$$

When $\frac{A^2}{2\sigma_2^2} \gg 1$,

$$\overline{E} \sim \frac{4}{3\pi} A^3 \tag{D.28}$$

In this case, the energy dissipation consumed by nonlinear damping is identical with that due to the sinusoidal exciting disturbance. When $\frac{A^2}{2\sigma_2^2} \ll 1$,

$$\overline{E} \sim \sqrt{\frac{8}{\pi}} \sigma_2^3 \tag{D.29}$$

This result coincides with the energy dissipation due to random exciting disturbance. The calculated result of Eq.(D.27) is shown in Fig.D.4. From this figure it is found that the energy dissipation due to two exciting disturbances is larger than that due to pure regular exciting disturbance, further than the sum of the energy dissipations due to regular and irregular exciting disturbances respectively.

Appendix E

Instantaneous p.d.f. of total second order response based on the Kac & Siegert theory

We shall consider a second order functional series as:

$$X(t) = \int_{\tau} g_1(\tau)\zeta(t-\tau)d\tau + \int_{\tau_1}\int_{\tau 2} g_2(\tau_1,\tau_2)\zeta(t-\tau_1)\zeta(t-\tau_2)d\tau_1 d\tau_2 \tag{E.1}$$

$$\equiv X^{(1)} + X^{(2)} \tag{E.2}$$

Now considering $\zeta(t)$ as transformed "**white noise**" process, and denoting by $q(t)$ the appropriate impulse response function of the linear filter giving $\zeta(t)$ from white noise process $W(t)$, it follows that

$$\zeta(t) = \int_{-\infty}^{\infty} q(\tau)W(t-\tau)d\tau \tag{E.3}$$

where $q(\tau)$ is a weighting function and $W(t)$ is a unit white noise which satisfies:

$$E[W(t)W(t-\tau)] = \delta(\tau) \tag{E.4}$$

Substituting $\zeta(t)$ as given by Eq.(E.3) into Eq.(E.1), we get the following relations:

$$X^{(1)}(t) = \int_{0}^{\infty} k_1(\tau)W(t-\tau)d\tau \tag{E.5}$$

where

$$k_1(\tau) = \int_0^\infty g_1(s)q(\tau - s)ds \tag{E.6}$$

and

$$X^{(2)}(t) = \int_0^\infty \int_0^\infty k_2(\tau_1, \tau_2)W(t - \tau_1)W(t - \tau_2)d\tau_1 d\tau_2 \tag{E.7}$$

where

$$k_2(\tau_1, \tau_2) = \int_0^\infty \int_0^\infty g_2(u, v)q(\tau_1 - u)q(\tau_2 - v)dudv \tag{E.8}$$

Since q, g_1, and g_2 should be filter functions with physical causality, they must vanish at infinity, and for practical purpose they may be considered zero outside a bounded region. Thus we shall consider T, which is sufficiently large, as the integral upper limit of the above equations.

Now, we shall consider the following integral equation :

$$\int_0^T k_2(x, y)\Lambda(y)dy = \lambda\Lambda(x) \tag{E.9}$$

Then this integral equation becomes the Fredholm type integral equation. Since $k_2(x, y)$ is symmetric kernels, it can be shown from the Fredholm type integral equation theory[6)] that

1) the eigenvalues and the corresponding eigenfunctions exist,

2) the eigenfunctions are mutually orthonormal,

3) the eigenvalues are all real,

4) the Mercer's theorem can be applied to express the positive semi -definite kernel as

$$k_2(x, y) = \sum_{i=1}^{\infty} \lambda_i \Lambda_i(x)\Lambda_i(y) \tag{E.10}$$

Substituting the above relation into the Eq.(E.7), we have:

$$X^{(2)}(t) = \sum_{i=1}^{\infty} \lambda_i [\int_0^\infty \Lambda_i(\tau)W(t - \tau)d\tau]^2 \tag{E.11}$$

If the stochastic process $W_i(t), i = 1, 2, \ldots$, are defined as

$$W_i(t) = \int_0^T W(t - \tau)\Lambda_i(\tau)d\tau \tag{E.12}$$

Eq.(E.11) becomes

$$X^{(2)}(t) = \sum_{i=1}^{\infty} \lambda_i W_i^2(t) \tag{E.13}$$

Furthermore from the relation (E.3) and the orthonormality of Λ_i , it can be seen that

$$E[W_i(t)W_j(t)] = 0 \quad \text{for } i \neq j \tag{E.14}$$

This means that $W_i(t)$ and $W_j(t)$ are uncorrelated random variables and therefore independent, since they are Gaussian, and that $E[W_i^2(t)] = 1, \{W_i\}$ is the family of the standard Gaussian random variables.

Similarly we expand the kernel k_1 in terms of the eigenfunctions $\{\Lambda_i\}$ as

$$k_1(\tau) = \sum_{i=1}^{\infty} c_i \Lambda_i(\tau) \tag{E.15}$$

where

$$c_i = \int_0^T k_1(\tau)\Lambda_i(\tau)d\tau \tag{E.16}$$

Then substituting Eq.(E.15) into Eq.(E.5) we have:

$$X^{(1)}(t) = \sum_{i=1}^{\infty} c_i W_i(t) \tag{E.17}$$

This leads to the following decomposition of the total second order response process:

$$X(t) = \sum_{i=1}^{\infty}(c_i W_i(t) + \lambda_i W_i^2(t)) \tag{E.18}$$

The instantaneous p.d.f. can be obtained from the inverse Fourier transform of its characteristic function. The characteristic function is defined by

$$\phi_X(\theta) = E[\exp(i\theta X)] = \prod_{j=1}^{\infty} E[\exp\{i\theta(c_j W_j + \lambda_j W_j^2)\}] \tag{E.19}$$

Since W_j have the p.d.f as

$$p_{W_j}(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) \tag{E.20}$$

by using the following identity:

$$\int_{-\infty}^{\infty} \exp(itx - \frac{ax^2}{2})dx = \sqrt{\frac{2\pi}{a}} \exp(-\frac{t^2}{2a}) \qquad \text{for } a > 0 \tag{E.21}$$

the characteristic function can be rewritten as

$$\phi_X(\theta) = \prod_{j=1}^{\infty} \frac{1}{\sqrt{1 - 2i\lambda_j\theta}} \exp[-\frac{c_j^2\theta^2}{2(1 - 2i\lambda_j\theta)}] \tag{E.22}$$

By the inverse Fourier transform of the characteristic function the instantaneous p.d.f. becomes:

$$p_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(\theta)\exp(-i\theta x)d\theta \tag{E.23}$$

Next we shall consider the integral equation (E.9). It can be simplified by defining

$$\psi_i(t) = \int_0^T q(u-t)\Lambda_i(u)du, \qquad 0 \le t \le T \tag{E.24}$$

Then the integral equation (E.9) can be rewritten as

$$\int_0^T \tilde{k}_2(t,u)\psi_i(u)du = \lambda_i\psi_i(t) \tag{E.25}$$

where

$$\tilde{k}_2(t,u) = \int_0^T R_\zeta(t-\tau)g_2(\tau,u)d\tau \tag{E.26}$$

The new set of eigenfunctions $\{\psi_i\}$ will satisfy the following normalization relation

$$\int_0^T\int_0^T g_2(\tau_1,\tau_2)\psi_i(\tau_1)\psi_j(\tau_2)d\tau_1 d\tau_2 = \lambda_i\delta_{ij} \tag{E.27}$$

where δ_{ij} is the Kronecker delta and the parameters c_i are determined by

$$c_i = \int_0^T g_1(\tau)\psi_i(\tau)d\tau \tag{E.28}$$

If the time domain kernel $g_2(\tau_1,\tau_2)$ is known, the integral equation may be solved to obtain eigenvalues and eigenfunctions. However, it seems to be more common to obtain these values and functions in frequency domain than to do in time domain. For this purpose we define the Fourier transform of $\psi_i(t)$ as

$$\hat{\Psi}_i(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_i(t)\exp(-i\omega t)dt \tag{E.29}$$

Then we obtain the frequency domain integral equation as follows:

$$\int_{-\infty}^{\infty} S_\zeta(\omega)G_2(\omega,-\nu)\hat{\Psi}_i(\nu)d\nu = \lambda_i\hat{\Psi}_i(\omega) \tag{E.30}$$

where $S_\zeta(\omega)$ is the two-side wave spectrum.
Equation (E.30) may be rewritten as

$$\int_{-\infty}^{\infty} K(\omega,\nu)\Psi_i(\nu)d\nu = \lambda_i\Psi_i(\omega) \tag{E.31}$$

where the kernel $K(\omega,\nu)$ is defined by

$$K(\omega,\nu)=\sqrt{S_\zeta(\omega)S_\zeta(\nu)}G_2(\omega,-\nu) \tag{E.32}$$

and the eigenfunctions $\Psi_i(\omega)$ by

$$\Psi_i(\omega)=\sqrt{S_\zeta(\omega)}\hat{\Psi}_i(\omega) \tag{E.33}$$

Since G_2 is symmetric, it follows that $K(\omega,\nu)=K(\nu,\omega)$, that is, $K(\omega,\nu)$ is the Hermitian. Since the eigenfunctions Λ_i are all real, $\Psi_i(-\omega)=\Psi_i^*(\omega)$ and the normalization condition is

$$\int_{-\infty}^{\infty}\Psi_i(\omega)\Psi_j^*(\omega)d\omega=\delta_{ij} \tag{E.34}$$

Equation (E.28) for c_i becomes

$$c_i=\int_{-\infty}^{\infty}G_1(\omega)\sqrt{S_\zeta(\omega)}\Psi_i^*(\omega)d\omega \tag{E.35}$$