Second-Order Wave Diffraction by an Axisymmetric Body

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Abstract

The main difficulty in calculation of third order force is its forcing on the free surface which includes second order potential and its spacial derivatives. Second order potential also obeys an inhomogeneous free-surface boundary condition, and its calculation needs to do an integration on the whole free surface. For third order calculation, its forcing term is needed on a big area on the free surface, and individual calculation of the second order potential at each point in the area is evidently not economic.

This report provides a detailed analysis for the second order diffraction of monochromatic waves by an axisymmetric body in finite water. For wave diffraction from a body of revolution with vertical axis, the report derives a new integral equation, which can cancel the leading singularity in the derivative of ring Green's functions automatically. For the second order potential, the report proposes a forward prediction method to calculate the integration on the free surface. By this method we only need to compute the infinite integration on the free surface directly for a few of points; then an one-step quadrature is applied successively outward from the body for potentials at other points. To get accurate results, different approaches are also used to deal with singularities in the ring Green's functions in the integration both on the body surface and free surface. The method has been implemented for body of revolution with vertical axes, but the theory is also available for arbitrary bodies.

Numerical examination is also made to validate the numerical code by comparing second order force and moment on uniform and truncated cylinders and second order diffraction potential on the free surface with some published results. The comparison shows that the present results have a good agreement with those results. At last, the method is used to compute the second order wave elevation around uniform and truncated cylinders.

Contents

1. Introduction

It was observed in model tests and prototype experiments that tension leg platforms (TLPs) and gravity based structures (GBS) experience sudden bursts of highly amplified resonant activities during storms. This phenomenon is called as the 'ringing'. The ringing phenomenon will induce extreme stress in tethers of TLP, and even induces tethers breaking. It was found that ringing occurs at low frequency non-breaking waves and ringing periods are about 3-5 times of the period of the corresponding incident waves. This means that third order force is an exciting source for ringing and its calculation will be significant in predicting ringing phenomenon.

Nonlinear problems are characterized by forcing term in their boundary conditions. For the second order potential, the forcing term on the free surface only includes first order potentials, which can be represented by some simple ways. However, the third order forcing term on the free surface includes both first and second order potentials. The difficulty in calculating third order forcing term is the complexity and time-consuming in the calculation of second order potential, which is needed on the whole free surface or a big area. Usually, an integral equation method is used to compute the second order velocity potential, in which integrations have to be carried out on both body and free surfaces.

To get rid of the potential at the considering order from the integration on the free surface and the sea bed, an oscillating source with corresponding frequency is usually applied as the Green's function. The basic representation of the Green's function is written in a function of the Bessel function of the first kind of zero order. Kriebel (1990) used this representation in calculation of second wave elevation around a uniform cylinder. After using Graf's addition theorem to represent the Bessel function by the radii of source and field points in a polar coordinate system located at origin of the cylinder, we can get an unanimous representation with multiplications of functions of the radii of field and source points, no matter the radius of field point is larger than the one of source point or not. Thus, we can integrate the forcing term on the free surface with the functions only relative to the radius of field point to get a wave number spectrum. Then, second order potential can be represented by the wave number spectrum in an explicit form. To calculate the third order forcing term on the free surface, Teng and Kato (1996) tried to use this method to compute second order diffraction potential from an axisymmetric body. They found that the wave number spectrum goes to infinite at a wave number of twice of incident wave number when water depth is not infinity. The reason is that there is a component with twice of incident wave number in the second order forcing term on the free surface, which is called as the 'locked wave' by Molin(1979). Multiplication of the second order forcing term with Bessel functions at that wave number will give constant contribution with the increase of distance. Applied some techniques to deal with the infinity, Teng and Kato (1996) found that it is still hard to get a good agreement with Eatock Taylor and Hung's (1987) on the second order forces on uniform cylinders. The reason is that the wave number spectrum from this method converges slowly at high wave number, especially its derivatives.

To compute second order potential and forces, Hunt and Baddour (1981) and Hunt and Williams (1982) applied Weber transformation method, which is similar as the method by applying the above mentioned Green's function. Later, it was found that their results are not reliable. It has been doubted that their velocity potential is incomplete. Recently, Newman (1996) studied the problem again by the same method. For second

order potential on body surface of a uniform cylinder, he used Wronskians and transformation of integration contour to overcome the numerical inefficiency. Then, he managed to get a good agreement with Eatock Taylor and Hung's on second order forces.

Another representation of Green's function is of the modified Bessel functions, which is gotten by transformation of integrating contour. The Green's function in this form converges quickly, but the problem is that it has different definitions when radius of source point is larger than the one of field point or vice versa. For computing the second order force on bodies, this representation does not give too much troubles. However, for second order potential on the free surface, the integration domain on the free surface has to be separated into two different ranges according to the radius of source point. Thus, the second order diffraction potentials at different positions can not be represented by an explicit representation. Chau and Eatock Taylor (1992) used a similar Green's function, which also satisfies the body surface condition, and developed a semi-analytic solution for uniform cylinder. They used this method to compute second order wave elevation in the near field surrounding the cylinder. Huang and Eatock Taylor (1996) even developed a semi-analytic solution for truncated cylinders. For third order calculation, second order potential is needed on the whole free surface or in a very big domain. Calculation by this method directly seems very expansive, as an infinite integration has to be carried out for each point. Malenica and Molin (1995) made some improvement on this method in their third order calculation. They applied a forward moving approach to predict the integration associated with Hankel function from smaller radius to bigger one step by step. But for the parts associated with an infinite summation of the modified Bessel functions, they still used the direct integration method as Chau and Eatock Taylor did.

The present work proposes an one-step forward prediction method for calculating the terms associated with modified Bessel functions. Special concerns are also paid on the treatment for the logarithmic singularity in the ring Green's functions. By this approach, the second order potential can be calculated much more efficiently in a big area, like to form the forcing term for third order problem. The method has been implemented for axisymmetric bodies, and no difficulty has been found for extending it to arbitrary bodies, like TLPs.

2. Free Surface Condition

We define a right-handed coordinate system (x, y, z), with origin at the center of the body, $z=0$ on the still free surface and the z-axis pointing upward (see Figure 1). The fluid is assumed to be homogenous and incompressible, and irrotational. There exists a velocity potential that satisfies the Laplace equation and the nonlinear free-surface boundary condition on

$$\Phi_{tt} + g\,\Phi_z + \frac{\partial}{\partial t}[\nabla\Phi\cdot\nabla\Phi] + \frac{1}{2}\nabla\Phi\cdot\nabla(\nabla\Phi\cdot\nabla\Phi) = 0 \tag{1}$$

the free surface $z = \zeta\,(x,y,t)$, defined by

$$\zeta = \frac{1}{g}\Phi_t - \frac{1}{2g}[\nabla\Phi\cdot\nabla\Phi] \tag{2}$$
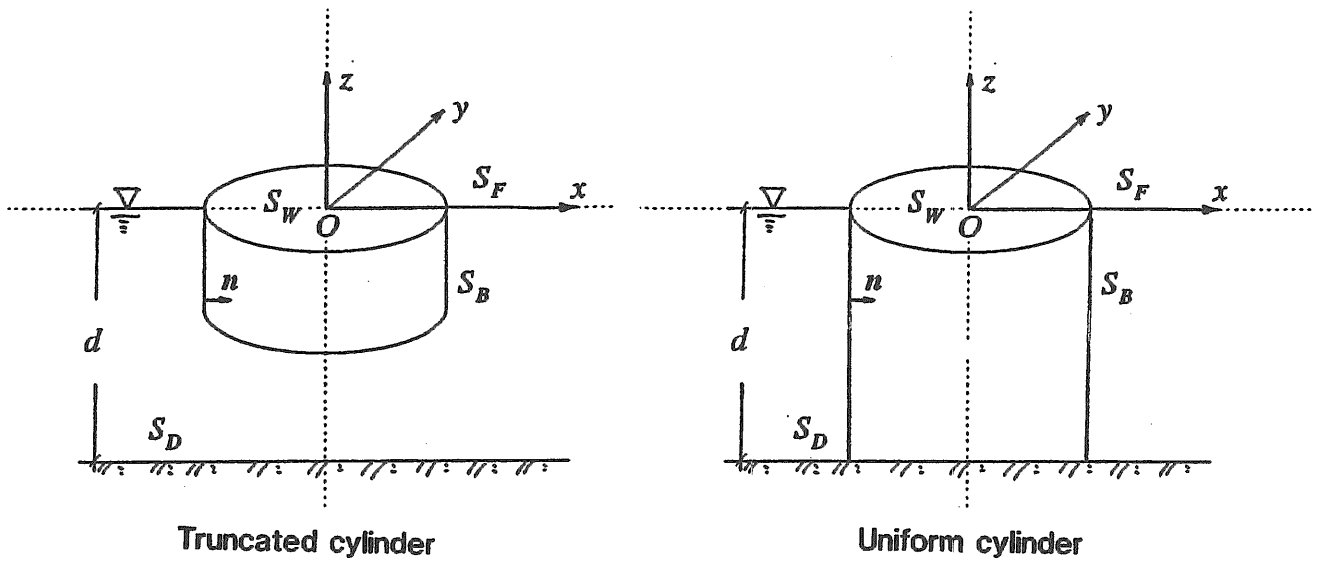
(4)

Figure 1. Definition sketch

Under the assumption of weak non-linearity, we can write the wave velocity potential as a perturbation series with respect to wave slope parameter $\varepsilon = kA$

$$\Phi = \varepsilon \, \Phi^{(1)} + \varepsilon^2 \, \Phi^{(2)} + \varepsilon^3 \, \Phi^{(3)} + \dots \qquad (3)$$

We assume that the incident monochromatic waves have an incident frequency ω. To solve the ringing phenomena, the first, second and third order harmonic potentials with the frequencies of $\omega_1 = \omega$, $\omega_2 = 2\omega$ and $\omega_3 = 3\omega$ are only considered. We separate the time dependencies explicitly, and write potentials at each order of ε as

$$\Phi^{(j)}(x,y,z,t) = Re[\phi^{(j)}(x,y,z)e^{-i\omega_j t}] \qquad (4)$$

After expanding eq. (1) into a perturbation series and collecting terms at the same order of ε, we can write the free surface conditions for the velocity potentials at each order of ε as

$$v_j \phi^{(j)} + \phi_z^{(j)} = q^{(j)} \qquad on \ z = 0 \qquad (5)$$

where

$$v_j = \omega_j^2 / g \qquad (6)$$

and the forcing terms at each order of ε

$$q^{(1)} = 0$$

$$q^{(2)} = -\frac{i\omega}{2g}\phi^{(1)}\left(-\frac{\omega^2}{g}\phi_z^{(1)} + \phi_{zz}^{(1)}\right) + \frac{i\omega}{g}\nabla\phi^{(1)}\cdot\nabla\phi^{(1)}$$

$$q^{(3)} = \frac{3i\omega}{g}\nabla\phi^{(1)}\cdot\nabla\phi^{(2)} - \frac{i\omega}{2g}\phi^{(1)}(\phi_{zz}^{(2)} - 4v\,\phi_z^{(2)}) - \frac{i\omega}{g}\phi^{(2)}(\phi_{zz}^{(1)} - v\,\phi_z^{(1)})$$

$$-\frac{1}{8g}\nabla\phi^{(1)}\cdot\nabla(\nabla\phi^{(1)}\cdot\nabla\phi^{(1)}) - \frac{v}{g}\phi^{(1)}\nabla\phi^{(1)}\cdot\nabla\phi_z^{(1)}$$

$$+\frac{1}{g}\left(\frac{v}{4}\phi^{(1)}\phi_z^{(1)} + \frac{1}{8}\nabla\phi^{(1)}\cdot\nabla\phi^{(1)}\right)(\phi_{zz}^{(1)} - v\,\phi_z^{(1)})$$

(7)

It can be seen that the third order forcing term includes both first and second order potential.

3. Integral Equation

For convenience in numerical calculation, we separate velocity potential into incident and diffraction potentials

$$\phi^{(j)} = \phi_I^{(j)} + \phi_D^{(j)}$$

(8)

By applying an oscillating source with frequency ω_j as the Green's function, we can obtain an integral equation for jth order diffraction potential as

$$\alpha\,\phi_D^{(j)}(x_0) - \iint_{S_B} \frac{\partial G(x;x_0;\omega_j)}{\partial n}\,\phi_D^{(j)}(x)\,ds$$

$$= \iint_{S_B} G(x;x_0;\omega_j)\frac{\partial\phi_I^{(j)}}{\partial n}\,ds - \iint_{S_F} G(x;x_0;\omega_j)\,q_D^{(j)}\,ds$$

(9)

where S_B and S_F denote the body and free surface, and $q_D^{(j)}$ is the difference between the total forcing term and the forcing one for incident waves

$$q_D^{(j)} = q^{(j)} - q_I^{(j)}$$

(10)

The positive direction of the normal to the body surface is defined as being out of the fluid. To weaken the singularity in the integration of derivative of the Green's function, we add another equation obtained inside the body (Eatock Taylor and Chau, 1992, and Teng and Eatock Taylor, 1995) and get a new equation as

$$(1 - v_j \iint_{S_W} G ds) \phi_D^{(i)}(x_0) + \iint_{S_B} \frac{\partial G}{\partial n} (\phi_D^{(i)}(x_0) - \phi_D^{(i)}(x)) ds$$

$$= \iint_{S_B} G \frac{\partial \phi_I^{(i)}(x)}{\partial n} ds - \iint_{S_F} G q_D^{(i)} ds$$

(11)

where S_W is the inner free surface. For axisymmetric body, we expand the velocity potential and the Green's function into series

$$\phi_D^{(i)}(x_0) = \sum_{m=0}^{\infty} \varepsilon_m \phi_{Dm}^{(i)}(r_0) cosm\theta_0 \qquad \phi_D^{(i)}(x) = \sum_{m=0}^{\infty} \varepsilon_m \phi_{Dm}^{(i)}(r) cosm\theta$$

(12)

$$G(x; x_0) = \sum_{m=0}^{\infty} \varepsilon_m G_m(r,z; r_0,z_0) cosm(\theta - \theta_0)$$

where ε_m is the coefficient of Neumann's polynomial($=1$ when $m=0$, 2 when $m>0$), e.g. see Watson,1966. Then, the integral equation for the mth mode in azimuthal angle θ of jth order potential can be obtained as

$$[\frac{1}{2\pi} - v_j \int_{\Gamma_W} G_0 r dr] \phi_{Dm}^{(i)}(r_0) - \int_{\Gamma_B} [\frac{\partial G_0}{\partial n} \phi_{Dm}^{(i)}(r_0) - \frac{\partial G_m}{\partial n} \phi_{Dm}^{(i)}(r)] r dl$$

$$= \int_{\Gamma_B} G_m \frac{\partial \phi_{Im}^{(i)}}{\partial n} r dl + \int_a^{\infty} G_m q_{Dm}^{(i)}(r) r dr$$

(13)

where Γ_B and Γ_W are the traces of S_B and S_W (see Figure 2), and the ring Green's function G_m is
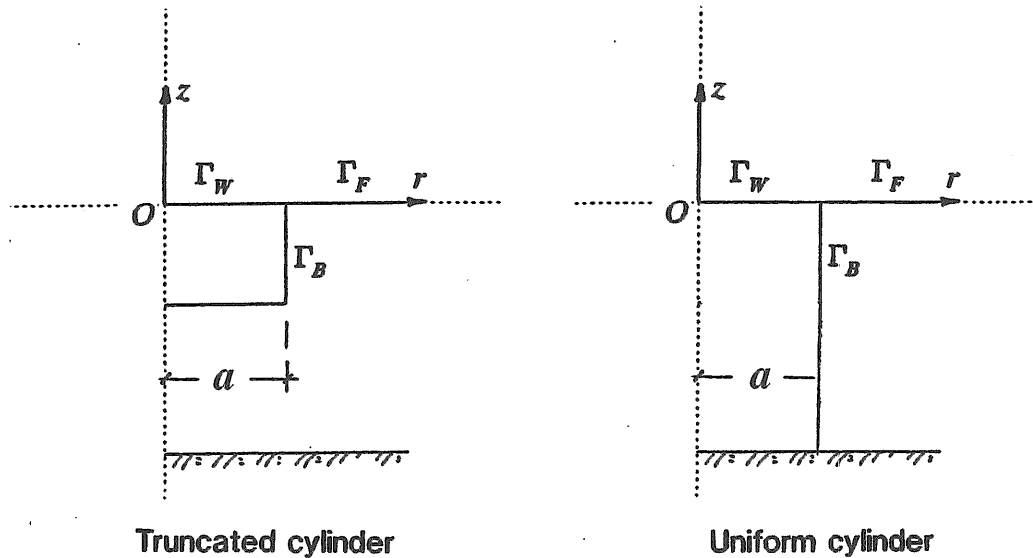

Figure 2. Integrating contour for axisymmetric body

$$G_m = -\frac{i}{2}C_0 H_m(k_j r_>) J_m(k_j r_<) Z_0(k_j z) Z_0(k_j z_0)$$

$$-\frac{1}{\pi}\sum_{n=1}^{\infty} C_n K_m(\kappa_{jn} r_>) I_m(\kappa_{jn} r_<) Z_n(\kappa_{jn} z) Z_n(\kappa_{jn} z_0) \qquad where \quad r_> > r_< \tag{14}$$

k_j and κ_{jn} are defined by dispersion equations of

$$\nu_j = k_j \tanh(k_j d) \qquad\qquad \nu_j = -\kappa_{jn} tg(\kappa_{jn} d) \tag{15}$$

where d is the water depth. The eigenfunctions in z-direction are defined by

$$Z_0(k_j z) = \frac{\cosh k_j(z+d)}{\cosh k_j d}, \qquad\qquad Z_n(\kappa_{jn} z) = \frac{\cos\kappa_{jn}(z+d)}{\cos\kappa_{jn} d} \tag{16}$$

and the factor C_0 and C_n are

$$C_0 = [2\int_{-d}^{0} Z_0^2(k_j z) dz]^{-1}, \qquad\qquad C_n = [2\int_{-d}^{0} Z_n^2(\kappa_{jn} z) dz]^{-1} \tag{17}$$

The ring-source potential and its derivative have been investigated by a number of researchers (Fenton, 1978; Hulme, 1983; Kim and Yue, 1989). It was found that the ring sources at each mode have the same logarithmic singularity when field point is close to the source point; and the leading singularities in their derivatives have the same form as the reverse of the distance between field point and source point. Thus, the leading term of singularity in equation (13) can be canceled each other. Other weak singularities are dealt with by suitable coordinate transformation (Telles, 1987).

For the second order potential, if the minimum radius of free surface is larger than or equal to the maximum radius of the body (otherwise, we should divide the free surface into different ranges), the integral on the free surface can be written as

$$I_F(r_0, \theta_0, z_0) = -\int_a^{\infty} r dr \sum_{m=0}^{\infty} \varepsilon_m q_{Dm}^{(2)}(r)\cos m\theta_0 \tag{18}$$

$$[\frac{i\pi}{2}C_0 H_m(k_2 r_>) J_m(k_2 r_<) Z_0(k_2 z_0) + \sum_{n=1}^{\infty} C_n K_m(\kappa_n r_>) I_m(\kappa_n r_<) Z_n(\kappa_n z_0)]$$

where $\kappa_n = \kappa_{2n}$ for brevity. By defining

$$S_{1m0}(a) = \frac{i\pi}{2}\int_a^{\infty} q_{Dm}^{(2)}(r) H_m(k_2 r) r dr, \qquad S_{1mn}(a) = \int_a^{\infty} q_{Dm}^{(2)}(r) K_m(\kappa_n r) r dr, \tag{19}$$

we can write the above integral equation as

$$I_F(r_0, \theta_0, z_0) = - \sum_{m=0}^{\infty} \varepsilon_m [C_0 S_{1m0}(a) J_m(k_2 r_0) Z_0(k_2 z_0) \tag{20}$$

$$+ \sum_{n=1}^{\infty} C_n S_{1mn}(a) I_m(\kappa_n r_0) Z_n(\kappa_n z_0)] \cos m\theta_0$$

Thus, for the second order potential on the body surface, integration on the free surface is only needed to run once.

The infinite integration of S_{1mn} $(n>0)$ converges quickly with the increase of the integration range as the modified Bessel function K_m decays at an exponential rate. However, the infinite integration of S_{1m0} is oscillating and converges slowly with the increase of integrating distance. A method widely used for its calculation is to divide the integration range into two parts. In the inner domain, a direct quadrature is used, and in the outer domain an analytic method is used to integrate it to infinity, after some asymptotic approximations have been used for Hankel functions.

4. Second Order Potential on Free Surface

For the second order potential at a point not close to the body, the following integral equation can be used to compute second order potential directly

$$\phi_D^{(2)}(x_0,0) - \iint_{S_B} \frac{\partial G}{\partial n} \phi_D^{(2)}(x,z) ds = \iint_{S_B} G \frac{\partial \phi_I^{(2)}(x,z)}{\partial n} ds - \iint_{S_F} G q_D^{(2)} ds \tag{21}$$

For axisymmetric body, the integral equation for mth mode with respect to azimuthal angle θ can be written as

$$\frac{1}{2\pi} \phi_{Dm}^{(2)}(r_0,0) = \int_{\Gamma_B} \frac{\partial G_m}{\partial n} \phi_{Dm}^{(2)}(r,z) \, r dl + \int_{\Gamma_B} G_m \frac{\partial \phi_{Im}(r,z)}{\partial n} r dl - \int_a^{\infty} G_m q_{Dm}^{(2)} r dr \tag{22}$$

However, when the computed point is close to the body surface, there are some quasi- singularities in the body integration. Direct use of the above integral equation will not give accurate results. For weakening the quasi-singularities, we add another integral equation

$$- \nu_2 \iint_{S_W} G(x,x_0) ds \phi_D(x^*) + \iint_{S_B} \frac{\partial G(x,x_0)}{\partial n} \phi_D^{(2)}(x^*) ds = 0 \tag{23}$$

to the above equation, where

$$x^* = a \cos\theta_0 \, , \qquad\qquad for \quad x_0 = r_0 \cos\theta_0 \tag{24}$$

is the closest point to the computed point on the water line. It yields a new integral equation

$$\phi_D^{(2)}(x_0) - v_2 \iint_{S_W} G ds \, \phi_D^{(2)}(x^*) + \iint_{S_B} \frac{\partial G}{\partial n}(\phi_D^{(2)}(x^*) - \phi_D^{(2)}(x)) ds$$

$$= \iint_{S_B} G \frac{\partial \phi_I^{(2)}(x)}{\partial n} ds - \iint_{S_F} G q_D^{(2)} ds \tag{25}$$

We expand the potentials into series

$$\phi_D^{(2)}(x_0) = \sum_{m=0}^{\infty} \varepsilon_m \phi_{Dm}^{(2)}(r_0) cos m\theta_0, \qquad \phi_D^{(2)}(x^*) = \sum_{m=0}^{\infty} \varepsilon_m \phi_{Dm}^{(2)}(a) cos m\theta_0, \tag{26}$$

Then, another integral equation can be obtained for the mth mode of second order potential as

$$\frac{1}{2\pi} \phi_{Dm}^{(2)}(r_0) = v_2 \int_{\Gamma_W} G_0 r dr \, \phi_{Dm}^{(2)}(a) - \int_{\Gamma_B} [\frac{\partial G_0}{\partial n} \phi_{Dm}^{(2)}(a) - \frac{\partial G_m}{\partial n} \phi_{Dm}^{(2)}(r)] r dl$$

$$+ \int_{\Gamma_B} G_m \frac{\partial \phi_{Im}^{(2)}}{\partial n} r dl - \int_a^{\infty} G_m q_{Dm}^{(2)} r dr \tag{27}$$

This equation will weaken the leading term of quasi-singularity, and the other quasi- singularities will still be dealt with by coordinate transformation.

5. Numerical Implementation

For second order potential on the body surface and in the fluid domain, two integrations have to be carried out both on body surface and free surface when an integral equation method is used. Direct calculation of those integrations is very expansive when second order potential is needed in a big area, like to form the third order forcing term on the free surface. Thus, some techniques have to be applied to aim at a speedup of the calculation. Besides the inefficiency in calculation of the integration on the free surface, the singularity in the ring Green's function has to be dealt with carefully. The ring Green's function is represented by an infinite summation of modified Bessel functions, and simple truncation will induce great inaccuracy when field point approaches to the source point. To calculated those integration accurately, some special approaches are used.

5.1 The integration on the free surface

Substituting the ring Green's function into the above integration, the integration on the free surface can be written as

$$I_{Fm}(r_0, \theta_0, 0) = - \int_a^{\infty} r dr \, q_{Dm}^{(2)}(r) [i\pi C_0 H_m(k_2 r_>) J_m(k_2 r_<) \tag{28}$$

$$+ 2 \sum_{n=1}^{\infty} C_n K_m(\kappa_n r_>) I_m(\kappa_n r_<)]$$