The Green's function is represented by multiplies of Bessel function and an infinite summation of modified Bessel functions. In numerical calculation the infinite summation has to be truncated at a big value N. As pointed out by Fenton (1978), there is a logarithmic singularity in the ring Green's function, and simple truncation will induce some inaccuracy. We define each term of the multiplication of modified Bessel function as

$$L_{mn} = 2C_n K_m(\kappa_n r_>) I_m(\kappa_n r_<) Z_n(\kappa_n z) \tag{29}$$

The limitation of L_{mn} term at large n is

$$L_{mn} \to L_{mn}^s = \frac{1}{n\pi}\frac{1}{\sqrt{rr_0}}\exp\left(-\frac{n\pi}{d}|r-r_0|\right)\cos\frac{n\pi z}{d} \tag{30}$$

and the infinite sum of L_{mn}^s can be written as

$$\begin{aligned} L_m^s &= \sum_{n=1}^{\infty} L_{mn}^s = \sum_{n=1}^{\infty}\frac{1}{n\pi}\frac{1}{\sqrt{rr_0}}\exp\left(-\frac{n\pi}{d}|r-r_0|\right) \\ &= \frac{-1}{2\pi\sqrt{rr_0}}\ln\left[1-2\exp\left(-\frac{\pi}{d}|r-r_0|\right)\cos\frac{\pi z}{d}+\exp\left(-\frac{2\pi}{d}|r-r_0|\right)\right] \end{aligned} \tag{31}$$

When field point is close to the source point, we expend the exponential and cosine functions into Taylor series and can find that it is a logarithmic singularity of

$$L_m^s \simeq \frac{-1}{2\pi\sqrt{rr_0}}\ln\left[\frac{\pi^2}{d^2}(z^2+(r-r_0)^2)\right] \tag{32}$$

To remove the logarithmic singularity, we write the infinite integration of modified Bessel function as

$$\begin{aligned} \sum_{n=1}^{\infty}\int_a^{\infty} L_{mn}(r)q_{Dm}^{(2)}(r)rdr &= \sum_{n=1}^{\infty}\int_a^{\infty} L_{mn}(r)q_{Dm}^{(2)}(r)rdr - \sum_{n=1}^{\infty}\int_{r_0-\Delta r_1}^{r_0+\Delta r_2} L_{mn}^s(r)q_{Dm}^{(2)}(r)rdr \\ &+ \int_{r_0-\Delta r_1}^{r_0+\Delta r_2} L_m^s(r)q_{Dm}^{(2)}(r)rdr \end{aligned} \tag{33}$$

Outside the range $(r-\Delta r_1,\ r+\Delta r_2)$, the higher modes in the first integration decay quickly and the summation can be truncated at a big value N; inside the range $(r-\Delta r_1,\ r+\Delta r_2)$ the integrand in the first two terms will be close to and cancel each other for big modes. Truncation at the big N will also give a good approximation. Thus, we can write the integration as

$$\sum_{n=1}^{\infty}\int_a^{\infty} L_{mn}(r)q_{Dm}^{(2)}(r)rdr \approx \sum_{n=1}^{N}\int_a^{\infty} L_{mn}(r)q_{Dm}^{(2)}(r)rdr - \sum_{n=1}^{N}\int_{r_0-\Delta r_1}^{r_0+\Delta r_2} L_{mn}^{s}(r)q_{Dm}^{(2)}(r)rdr + \int_{r_0-\Delta r_1}^{r_0+\Delta r_2} L_{m}^{s}(r)q_{Dm}^{(2)}(r)rdr \tag{34}$$

For the third term, a coordinate transform

$$r=\frac{1}{2}\Delta r\xi+\frac{1}{2}(2r_0+\Delta r), \qquad \xi=\frac{1}{2}\eta^2+\eta-\frac{1}{2}, \qquad dr=\frac{1}{2}\Delta r(\eta+1) \tag{35}$$

is used to remove the singularity. Then, we represent the integration on the free surface as

$$I_{Fm}(r_0,0)=-C_0(S_{1m0}(r_0)J_m(k_2r_0)+S_{2m0}(r_0)H_m(k_2r_0)) - \sum_{n=1}^{N}C_n(S_{1mn}(r_0)e^{-\kappa_n r_0}I_m(\kappa_n r_0)+S_{2mn}(r_0)e^{\kappa_n r_0}K_m(\kappa_n r_0))+\sum_{n=1}^{N}V_{mn}(r_0)-V_{m0}(r_0) \tag{36}$$

The functions in the above equation are defined by

$$S_{1m0}(r_0)=\frac{i\pi}{2}\int_{r_0}^{\infty} q_{Dm}^{(2)}(r)H_m(k_2r)rdr, \qquad S_{1mn}(r_0)=\int_{r_0}^{\infty} q_{Dm}^{(2)}(r)K_m(\kappa_n r)rdr\,e^{\kappa_n r_0},$$
$$S_{2m0}(r_0)=\frac{i\pi}{2}\int_{a}^{r_0} q_{Dm}^{(2)}(r)J_m(k_2r)rdr, \qquad S_{2mn}(r_0)=\int_{a}^{r_0} q_{Dm}^{(2)}(r)I_m(\kappa_n r)rdr\,e^{-\kappa_n r_0}. \tag{37}$$

and

$$V_{m0}(r_0)=-\frac{1}{2\pi}\int_{r_0-\Delta r_1}^{r_0+\Delta r_2} q_{Dm}^{(2)}(r)\ln[1-\exp(-\frac{\pi}{d}|r-r_0|)]\sqrt{\frac{r}{r_0}}dr$$
$$V_{mn}(r_0)=\frac{1}{2n\pi}\int_{r_0-\Delta r_1}^{r_0+\Delta r_2} q_{Dm}^{(2)}(r)\exp(-\frac{\pi}{nd}|r-r_0|)\sqrt{\frac{r}{r_0}}dr \tag{38}$$

Here, the exponential functions are applied to multiply with modified Bessel functions to guarantee the integrand not to overflow from both upper and lower limits of computers at large variable.

The integration in the function V_{mn} is localized in a small range $(r-\Delta r_1, r+\Delta r_2)$, and can be calculated easily. However, the integrations in S_{1mn} and S_{2mn} have to be carried out from the body to infinity. Direct calculation will be very expansive, especially when calculation for many points is needed. Here, a one-step forward prediction method is applied. By this method, after we have gotten those functions at a point r_0, the functions at next point r_1, which is close to r_0, can be obtained by

$$S_{1m0}(r_1) = \frac{i\pi}{2}\int_{r_1}^{\infty} q_{Dm}^{(2)}(r)H_m(k_2 r)\,r\,dr = S_{1m0}(r_0) - \frac{i\pi}{2}\int_{r_0}^{r_1} q_{Dm}^{(2)}(r)H_m(k_2 r)\,r\,dr$$
$$S_{2m0}(r_1) = \frac{i\pi}{2}\int_{a}^{r_1} q_{Dm}^{(2)}(r)J_m(k_2 r)\,r\,dr = S_{2m0}(r_0) + \frac{i\pi}{2}\int_{r_0}^{r_1} q_{Dm}^{(2)}(r)J_m(k_2 r)\,r\,dr \tag{39}$$

and

$$S_{1mn}(r_1) = \int_{r_1}^{\infty} q_{Dm}^{(2)}(r)K_m(\kappa_n r)\,r\,dr = S_{1mn}(r_0)e^{\kappa_n(r_1-r_0)} - \int_{r_0}^{r_1} q_{Dm}^{(2)}(r)K_m(\kappa_n r)\,r\,dr\,e^{\kappa_n r_1}$$
$$S_{2mn}(r_1) = \int_{a}^{r_1} q_{Dm}^{(2)}(r)J_m(\kappa_n r)\,r\,dr = S_{2mn}(r_0)e^{-\kappa_n(r_1-r_n)} + \int_{r_0}^{r_1} q_{Dm}^{(2)}(r)I_m(\kappa_n r)\,r\,dr\,e^{-\kappa_n r_1} \tag{40}$$

This method can save a lot of computing effort. But after a number of steps, the prediction will lose accuracy, and even diverse. A remedy to solve this problem is just to use the direct integration method to compute them again before a tolerant error has been accumulated. The number of steps used in forward prediction will be determined by a series forehead numerical examination, and is different for different modes.

For the integrations of Hankel and Bessel functions, the terms called as S_{1m0} and S_{2m0} in this paper, Malenica (1994) used a similar method to deal with them. But for the infinite summation of modified Bessel functions, S_{1mn} and S_{2mn} in this paper, Malenica still used the direct integration method to compute them. Another distinct difference between the present method and Malenica's is the approach used to treat the singularity in the ring Green's function. The present method only adds an auxiliary integration in a small adjacent area of the source point, but Malenica's is to add an auxiliary integration in the same area as the original integration.

5.2 The integration on the body surface

For those points not close to body surface, we apply equation (21) to compute the second order potential. The integration on the body surface can be written as

$$I_{Bm}(r_0,0) = \int_{\Gamma_B} [\phi_{Dm}^{(2)}(r,z)\,\frac{\partial G_m(r,z;r_0,0)}{\partial n} + \frac{\partial \phi_{Im}^{(2)}(r,z)}{\partial n}\,G_m(r,z;r_0,0)]\,r\,dl \tag{41}$$

After integrating with respect to the coordinates of field point on the body surface, the body surface integration can be simply written as

$$I_{Bm}(r_0,0) = U_{m0}H_m(k_2 r_0) + \sum_{n=1}^{\infty} U_{mn}K_m(\kappa_n r_0) \tag{42}$$

where

$$U_{m0} = -\frac{iC_0}{2}\int_{\Gamma_B}[\phi^{(2)}_{Dm}(r,z)\,(\frac{\partial J_m(k_2r)}{\partial r}n_r Z_0(k_2z)+J_m(k_2r)\frac{\partial Z_0(k_2z)}{\partial z}n_z)$$

$$+\frac{\partial\phi^{(2)}_{Im}(r,z)}{\partial n}J_m(k_2r)Z_0(k_2z)]\,r\,dl \qquad (43)$$

$$U_{mn} = -\frac{C_n}{\pi}\int_{\Gamma_B}[\phi^{(2)}_{Dm}(r,z)\,(\frac{\partial I_m(\kappa_n r)}{\partial r}n_r Z_n(\kappa_n z)+I_m(\kappa_n r)\frac{\partial Z_n(\kappa_n z)}{\partial z}n_z)$$

$$+\frac{\partial\phi^{(2)}_{Im}(r,z)}{\partial n}I_m(\kappa_n r)Z_n(\kappa_n z)]\,r\,dl$$

For those points close to body surface, a direct integration method is performed with an application of equation (27).

6. Numerical Results

6.1 Second order force

To ensure the numerical code reliable, we first make a comparison on the second order force with some published results.

After we have resolved the second order potential, the second order wave force and moment can be obtained by direct integration of hydrodynamic pressure on the body surface with an approximation to second order in ε. After some mathematical derivation, the equation of the second order generalized force on a vertically piercing fixed body in monochromatic waves can be written as

$$F_T^{(2)} = Re[(f_p^{(2)}+f_q^{(2)})e^{-2i\omega t}] \qquad (44)$$

where

$$f_p^{(2)} = \frac{\rho}{2}\iint_{S_B} i\omega\,(\phi_I^{(2)}+\phi_D^{(2)})n\,ds \qquad (45)$$

is the contribution from the second order velocity potentials; and is the contribution from quadratic multiplication of first order velocity potentials and height of water surface ζ, where $n=(n_r,\ n_z,\ (z-z_c)n_r-rn_z)$, z_c is the center of mass and C_W is the water line.

For axisymmetric body, we rearrange the quadratic multiplication of first order terms as

$$\nabla\phi^{(1)}\cdot\nabla\phi^{(1)} = \sum_{m=0}^{\infty}\varepsilon_m\varphi_m\cos m\theta \qquad \zeta^{(1)}\zeta^{(1)} = \sum_{m=0}^{\infty}\varepsilon_m\eta_m\cos m\theta \qquad (47)$$

and write the second order surge force as

$$f_{qx}^{(2)} = -\frac{\pi}{2}\int_{\Gamma_B} \rho\, \varphi_1 n_r r dl + \frac{\pi}{2} \rho g a\, \eta_1 n_r \Big|_{R=a}$$
$$f_{px}^{(2)} = -\frac{\pi}{2}\int_{\Gamma_B} \rho\, \phi_1^{(2)} n_r r dl \tag{48}$$

the second order heave force as

$$f_{qz}^{(2)} = -\frac{\pi}{2}\int_{\Gamma_B} \rho\, \varphi_0 n_z r dl + \frac{\pi}{2} \rho g a\, \eta_0 n_z \Big|_{R=a}$$
$$f_{pz}^{(2)} = -\frac{\pi}{2}\int_{\Gamma_B} \rho\, \phi_0^{(2)} n_z r dl \tag{49}$$

and the second order pitch moment as

$$m_{qy}^{(2)} = -\frac{\pi}{2}\int_{\Gamma_B} \rho\, \varphi_1 n_r r dl + \frac{\pi}{2} \rho g a\, \eta_1 n_r \Big|_{R=a}$$
$$m_{py}^{(2)} = -\frac{\pi}{2}\int_{\Gamma_B} \rho\, \phi_1^{(2)} n_r r dl \tag{50}$$

Tables 1-3 are comparison with Eatock Taylor and Hung's (1987) results on the part of second order force from second order potential, $f_p^{(2)}$, on uniform cylinders in different water depth d/a= 1.0, 3.0 and 10.0. The Eatock Taylor and Hung's results have been rearranged by summing up of the data in the last three columns of their paper. The meshes used in the present calculation are 4, 16 and 40 elements respectively for the three cylinders. It can be seen that for all of the three cylinders and in the all frequency range of the calculation, the present results have a good agreement with Eatock Taylor and Hung's.

Table 1. Comparison on f_p, the part of second order forces from second potential, on a uniform cylinder h/a=1.0.

ka	Present Results		Ref. (1)	
	Re	Im	Re	Im
0.25	.7126E+05	-.3640E+06	.7144E+05	-.3649E+06
0.50	.4469E+05	-.9970E+05	.4477E+05	-.9992E+05
0.75	.1138E+05	-.3294E+05	.1140E+05	-.3300E+05
1.00	.8931E+04	-.1181E+05	.8937E+04	-.1183E+05
1.25	.1802E+05	-.3080E+04	.1804E+05	-.3038E+04
1.50	.2679E+05	-.1115E+04	.2687E+05	-.1116E+04

Table 2. Comparison on f_b, the part of second order forces from second potential, on a uniform cylinder h/a=3.0.

ka	Present Results Re	Im	Ref. (1) Re	Im
0.25	.1268E+05	–.3118E+05	.1272E+05	–.3133E+06
0.50	.6409E+04	.8404E+04	.6431E+04	.8417E+04
0.75	.8837E+04	.1819E+05	.8868E+04	.1824E+05
1.00	.1963E+05	.1961E+05	.1966E+05	.1979E+05
1.25	.3385E+05	.1585E+05	.3396E+05	.1597E+05
1.50	.4521E+05	.7071E+04	.4538E+05	.7098E+04

Table 4 is the second order pitch moment with respect to the free surface on those cylinders. For this case, we have not found available published results to compare with. From the table, it can be seen that the component from the second order potential is the dominant part. In shallow water, large moment occurs at low frequency; in deep water, large moment occurs at high frequency.

Table 3. Comparison on f_b, the part of second order forces from second potential, on a uniform cylinder h/a=10.0.

ka	Present Results Re	Im	Ref. (1) Re	Im
0.25	.7126E+05	–.3640E+06	.7144E+05	–.3649E+06
0.50	.4469E+05	–.9970E+05	.4477E+05	–.9992E+05
0.75	.1138E+05	–.3294E+05	.1140E+05	–.3300E+05
1.00	.8931E+04	–.1181E+05	.8937E+04	–.1183E+05
1.25	.1802E+05	–.3080E+04	.1804E+05	–.3038E+04
1.50	.2679E+05	–.1115E+04	.2687E+05	–.1116E+04

Table 4. Second order pitch moment $m_y^{(2)}/\rho g a^2 A^2$ (with respect to the free surface) on uniform cylinders.

h/a	*ka*	m_{yq} Re	m_{yq} Im	m_{yp} Re	m_{yp} Im	m_{yt} Re	m_{yt} Im
1.0	0.25	0.0097	-0.1906	-3.4751	17.7610	-3.4654	17.5710
	0.50	0.6410	-0.3113	-2.0256	4.5736	-1.9615	4.2623
	0.75	0.1346	-0.2991	-0.4948	1.3555	-0.3602	1.0564
	1.00	0.1696	-0.2075	-0.4648	0.3174	-0.2952	0.1100
	1.25	0.1711	-0.1125	-0.8715	-0.1044	-0.7004	-0.2169
	1.50	0.1456	-0.0493	-1.2153	-0.1432	-1.0697	-0.1925
3.0	0.25	0.0231	-0.3900	-1.4726	3.8020	-1.4496	3.4120
	0.50	0.0974	-0.2289	-0.5558	-1.0018	-0.4584	-1.2307
	0.75	0.1653	-0.0299	-1.0021	-2.3006	-0.8367	-2.3305
	1.00	0.2257	0.0202	-1.9792	-2.8090	-1.7535	-2.7889
	1.25	0.2580	0.0873	-3.1615	-2.5160	-2.9035	-2.5073
	1.50	0.2393	-0.0107	-4.1937	-1.3519	-3.9544	-1.3626
10.0	0.25	0.0214	0.1476	-0.6531	-1.7621	-0.6317	-1.6145
	0.50	0.0707	0.2423	-0.8894	-4.0820	-0.8187	-3.8397
	0.75	0.1520	0.1406	-1.5668	-6.4158	-1.4148	-6.2752
	1.00	0.2269	0.0632	-2.9981	-7.3396	-2.7711	-7.2764
	1.25	0.2610	0.0172	-5.3630	-6.2119	-5.1020	-6.1948
	1.50	0.2409	-0.0094	-8.0749	-3.1548	-7.8341	-3.1642

Tables 5 and 6 are the second order surge and heave forces on a truncated cylinder with radius *a* and draft $T/a=4$ in a water depth of $d/a=10$. A mesh of 25 elements on the body trace Γ_B is used in this calculation. Comparison is made with Huang and Eatock Taylor's (1996) recent results at a few available frequencies. It can be seen that the present results also have a good agreement with Huang and Eatock Taylor's results.

Table 5. Comparison on second order surge forces $f_x^{(2)}/\rho gaA^2$ on a truncated cylinder of radius a and draft $T/a=4$ in a water depth of $d/a=10$, first row is the part from the first order potential f_{xq}, the second row is the part from the second order potential f_{xp} and the last row is the total force f_{xt}.

ka	Present Results		Huang and Eatock Taylor's			
			Semi-analytical		BEM results	
	Re	Im	Re	Im	Re	Im
0.8	-0.4306	-1.4561	-0.4425	-1.4746	-0.4452	-1.4746
	1.0265	1.9767	1.0144	1.9740	1.0261	1.9702
	0.5960	0.5206	0.5719	0.4994	0.5808	0.4991
1.0	-0.8828	-1.1533				
	1.9253	1.9518				
	1.0425	0.7985				
1.2	-1.3653	-0.7721	-1.4356	-0.7970	-1.4334	-0.8024
	3.0557	1.6631	3.0406	1.6594	3.0451	1.6570
	1.6904	0.8910	1.6050	0.8629	1.6116	0.8552
1.4	-1.7242	-0.3821				
	4.0810	1.0968				
	2.3568	0.7146				
1.6	-1.8386	-0.0048				
	4.6876	0.2180				
	2.8490	0.2132				
1.8	-1.6679	0.3694				
	4.6072	-1.0689				
	2.9393	-0.6995				
2.0	-1.2595	0.7371	-1.3530	0.7652	-1.3629	0.7606
	3.9259	-2.4594	3.9052	-2.4883	3.8702	-2.4750
	2.6663	-1.7224	2.5529	-1.7233	2.5071	-1.7144

Table 7 is the second order pitch moment on the same truncated cylinder. The water depth and the mesh used are the same as for Tables 5 and 6, but calculation is run at different wave numbers for comparing with Table 4. It can be seen from the comparison between Tables 4 and 7 that the contributions from first order potential are almost the same for the uniform and truncated cylinders, especially at high frequency; but the contributions from the second order potential have a lot of difference. The reason is that the second order diffraction potential decays slowly with z-coordinate. It can also be seen that as for uniform cylinders the contribution from second order potential is the dominant part for this truncated cylinder.

Table 6. Comparison on second order total heave forces $f_z^{(2)}/\rho gaA^2$ on a truncated cylinder of radius a and draft $T/a=4$ in a water depth of $d/a=10$.

ka	Present Results		Huang and Eatock Taylor's			
			Semi-analytical		BEM results	
	Re	Im	Re	Im	Re	Im
0.8	0.0028	0.4075	0.0020	0.4047	0.0026	0.4084
1.0	0.0038	0.4113				
1.2	0.0691	0.3344	0.0690	0.3319	0.0695	0.3347
1.4	0.1908	0.2007				
1.6	0.3366	0.0445				
1.8	0.4538	-0.1037				
2.0	0.4952	-0.2301	0.4941	-0.2319	0.4960	-0.2331

Table 7. Second order pitch moment $m_y^{(2)}/\rho g a A^2$ (with respect to the free surface) on a truncated cylinder of radius a and draft T/a=4 in a water depth of d/a=10.

ka	m_{yq} Re	m_{yq} Im	m_{yp} Re	m_{yp} Im	m_{yt} Re	m_{yt} Im
0.25	0.0146	0.3363	-0.3198	-0.4347	-0.3052	-0.0984
0.50	0.0647	0.2712	-0.4676	-1.6493	-0.4028	-1.3781
0.75	0.1500	0.1462	-1.0222	-2.5751	-0.8722	-2.4289
1.00	0.2268	0.0634	-2.0075	-3.0066	-1.7808	-2.9432
1.25	0.2613	0.0168	-3.2300	-2.6530	-2.9688	-2.6362
1.50	0.2410	-0.0097	-4.3187	-1.4233	-4.0777	-1.4330

6.2 S_{1mn} and S_{2mn} functions

To accelerate the convergence of the integration on the free surface, this report suggested a forward prediction method for compute the functions S_1 and S_2 after having got the functions at a position. Before applying the method for practical calculation, some examinations on the forward prediction method have to be carried out. Here an examination is made on a uniform cylinder of radius a in a water depth of d/a=10. Wave number ka is 0.5. The examination is only made on the zero Fourier mode and *0*th, *1*st, *10*th and *50*th eigen-modes of the functions. The step length used in the forward prediction is 0.01257a.

Figure 3 shows the comparison of function S_1 by the direct integration and forward prediction method. For the zero eigen-mode, the forward prediction method is to compute the function at $r=a$ by the direct integration method, then to apply the forward prediction method successively. From Figs. 3-A and 3-B we can see that even though we have applied the forward prediction method for 2000 steps, the results are still the same with the one from the direct integration method. We can not see any difference between the results from the two different methods.